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PION OBSERVABLES IN THE EXTENDED NJL MODEL WITH VECTOR AND AXIAL-VECTOR MESONS[1]

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Abstract

The momentum-space bosonization method of a Nambu and Jona-Lasinio type model with vector and axial-vector mesons is applied to $\pi\pi$ scattering. Unlike the case in earlier published papers, we obtain the $\pi\pi$ scattering amplitude using the linear and nonlinear realizations of chiral symmetry and fully taking into account the momentum dependence of meson vertices. We show the full physical equivalence between these two approaches. The chiral expansion procedure in this model is discussed in detail. Chiral expansions of the quark mass, pion mass and constant f_π are obtained. The low-energy $\pi\pi$ phase shifts are compared to the available data. We also study the scalar form factor of the pion.

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1 Introduction

Coloured quarks and gluons are the main objects in the fundamental theory of the strong interactions, quantum chromodynamics (QCD), which are not directly observed in experiment. At intermediate and low energies the laws governing interactions of these fundamental particles bring about complex rearrangements inside a quark-gluon system which eventually result in two-quark and three-quark colourless formations—hadrons. A second transition appears at a comparable scale, namely the spontaneous chiral symmetry breakdown, leading to the pseudo-Goldstone boson octet. These phenomena cannot be consistently described within perturbation theory in the strong coupling constant α_s which is no longer small at the typical hadronic energy scale. Therefore, it is mandatory to develop nonperturbative methods to solve these problems.

There are few nonperturbative methods employed in particle physics. Noteworthy are lattice field theory, quantization on the light cone, and QCD sum rules. Despite all their advantages, these approaches have not yet lead to a unified description of confinement and spontaneous chiral symmetry breakdown (χSB). Both phenomena are supposed to arise from colour gauge interactions. Yet, the complex process of hadron formation remains obscure. This stimulates numerous attempts to construct models aimed at describing individual major features of the process.

One can find explanation of spontaneous symmetry breakdown in the Nambu–Jona-Lasinio model, chiral quark model, approaches with stochastic vacuum fields, dual QCD, etc. A large class of such models involve path integral methods for calculation of Green functions that approximate QCD in the infrared region. The guiding principle is iterative integration first with respect to rapidly varying fields and then with respect to slowly varying ones, an idea borrowed from the general theory of phase transitions. According to this idea, slow modes mainly determine the asymptotic behaviour of the system at small energies and momenta (infrared asymptotics) and require thorough study in terms of collective variables. The contribution from "fast" fields is less substantial and can be taken into account by perturbation theory. The Nambu–Jona-Lasinio model (NJL) [1], on which we concentrate below, is one of these schemes. The boundary between "slow" and "fast" fields is largely conventional being dictated by physical considerations. The chiral $SU(3)_L \otimes SU(3)_R$ symmetry of the QCD lagrangian is broken at energies of the order of $\Lambda_{\chi SB} \sim 1\,\mathrm{GeV}$ if small quark masses are ignored. This is what allows one to relate $\Lambda_{\chi SB}$ to the boundary between "large" and "small" momenta when looking into infrared QCD asymptotics from the symmetry point of view.

Chiral symmetry breakdown is a fundamental issue because it is related to the problem of elementary particle masses and chiral symmetry lends itself to a controlable expansion (in energy) at energies below $\Lambda_{\chi SB}$. At these energies the QCD vacuum becomes highly non-perturbative due to the condensation of quark-antiquark pairs as a consequence of the spontaneous symmetry violation. It is believed that any serious model of the strong interactions has to account for these facts. This is the case for the celebrated NJL model which was originally formulated not in terms of quark but rather nucleonic degrees of freedom. In what follows we will be concerned with a modern extension based on the

quark description.

Assuming that the NJL model in the form that involves all essential QCD symmetries may be a reasonable approximation of QCD in the intermediate energy range, we develop a new theoretical method. It allows arbitrary N-point functions describing strong and electroweak interactions of basic meson states to be calculated through the effective four-quark interaction of the NJL model [2]. Now there are two approaches to the problem. One is based on the classical treatment of four-fermion Lagrangians (the Hartree–Fock approximation plus Bethe – Salpeter type equations for bound states) [3]-[7]. The other involves explicit separation of collective degrees of freedom in the functional integral of the theory (bosonization) [8]-[14]. In this framework one usually derives the Lagrangian based on a derivative expansion instead of directly calculating the Green functions. For technical reasons, such a derivative expansion can only be performed to the first few orders. *We have constructed such a formal scheme that at the same time incorporates bosonization and has all advantages of the pure fermionic approach.* It allows to combine the advantages of both of the abovementioned methods. The results of our more general method can be directly compared with the ones based on standard bosonization by performing a chiral expansion based on the nonlinear Bethe – Salpeter equations.

Originally formulated in terms of boson variables linearly transforming with respect to the chiral group action, the method can be extended to nonlinear realization of symmetry. This is one of the significant advantages of the approach as compared with Hartree and Fock's classical technique, which is linear is this sense. Now chiral symmetry consequences are often investigated via nonlinear chiral Lagrangians. They are a series in energy (E), which is equivalent to normal expansion with an increasing number of derivatives. As a rule, investigations are mostly confined to the $\mathcal{O}(E^4)$ approximation. Recently a program like this was used for the extended NJL model [14]. In the present paper we show how one can generalize these results without any reference to an expansion in E^2.

Apart from pure theoretical developments to generalize the results of [14], we carry out explicit calculations. Our primary concern is the $\pi\pi$ scattering amplitude. We shall calculate it both in the linear and nonlinear approaches using the standard Lagrangian of the extended NJL model with scalar, pseudoscalar, vector, and axial-vector four-quark interactions. We show that there is no difference between these two approaches. The expressions for the total amplitude $A(s,t,u)$ are the same in both cases. It has to be so because the same Lagrangian is used. We consider this point in detail to show the self-consistency of our method.

The paper is organized as follows. In the next section we introduce the Lagrangian and notation as well as review the main steps of the momentum-space bosonization technique. We then consider the nonlinear realization of chiral symmetry. We shall show that in this case equations for masses of bound states coincide with analogous equations in the linear approach. The expression for the weak pion decay constant f_π does not change either. All this allows us to get common rules for constructing chiral expansions in the linear and nonlinear approaches. In section 3 we derive the $\pi\pi$ scattering amplitudes and the main low-energy characteristics of the process in question. Theoretical derivations of the previous section will be the starting point. At first we shall use the model with

linear realization of chiral symmetry. In this section our calculations are an extension of earlier investigations [2, 15]. The corresponding amplitude includes the exchange of scalar and vector mesons as well as quark box diagrams. We give explicit expressions for this amplitude in terms of a few standard integrals. We obtain the low-energy expansion for the scattering amplitude and recover Weinberg's [16] formula (to lowest order in the energy expansion). Examining the model arising from classification of collective excitations based on nonlinear representations of the chiral group, we arrive at the same expression for the $\pi\pi$ scattering amplitude. We demonstrate this point by direct calculations and comparing the different contributions. In this part of the paper we not only generalize the known result of [14], we also prove the equivalence of this approach with the other known results. Our method looks like a bridge between different NJL models. Finally we present our results, comparing different variants of the NJL model with experiment and chiral perturbation results. In section 4 we calculate the scalar form factor of the pion. We pay particular attention to its relation to the value of the constituent quark mass. We conclude with a summary of our results.

2 The theoretical background

In Ref.[17] the extended NJL model was investigated and it was shown how to calculate Fourier transforms of N-point Green functions by a simple method which was proposed in [2]. This calculation was done in the linear realization of chiral symmetry. In this section we are developing the method to apply it to the extended NJL model with nonlinear realization of chiral symmetry. To describe collective excitations we shall use induced representations of the chiral group. Calculations of quark loop diagrams will involve renormalization of collective variables, as is typical of our approach. As a result, we are able to calculate amplitudes of physical processes to any accuracy in E^2 in models with nonlinear realization of chiral symmetry.

2.1 The linear approach

In this section we summarize previous results (see also paper [17]). The reader who is familiar with the momentum-space bosonization method can use this section as a collection of our notation and main formulae to be found in this paper. The main idea of this method consists in the construction of special bosonic variables to be used for the description of the observable mesonic states. As a result, it extends the usual treatment of bosonized NJL models, which was formulated in [8] and developed in [9,10,12]. The standard approach is essentially linked to the derivative expansion of the effective meson Lagrangian. The momentum-space bosonization method does not use this approximation. It involves the bosonization procedure and has all the advantages of the pure fermionic approach (Hartree-Fock plus Bethe-Salpeter approximation). Instead of step-by-step construction of the effective Lagrangian describing the dynamics of collective excitations we develop a method that allows direct calculation of amplitudes for particular physical processes. The

amplitudes derived accumulate the entire information on the process under investigation, just as if we had a total effective meson Lagrangian of the bosonized NJL model (in the one-loop approximation).

Consider the extended $SU(2)_L \otimes SU(2)_R$ NJL Lagrangian with the local four-quark interactions

$$\begin{aligned} L(q) = \bar{q}(i\gamma^\mu\partial_\mu - \widehat{m})q \;\; &+ \;\; \frac{G_S}{2}\left[(\bar{q}\tau_a q)^2 + (\bar{q}i\gamma_5\tau_a q)^2\right] \\ &- \;\; \frac{G_V}{2}\left[(\bar{q}\gamma^\mu\tau_a q)^2 + (\bar{q}\gamma_5\gamma^\mu\tau_a q)^2\right], \end{aligned} \tag{1}$$

where $\bar{q} = (\bar{u}, \bar{d})$ are coloured ($N_c = 3$) current quark fields with current mass $\widehat{m} = \mathrm{diag}(\widehat{m}_u, \widehat{m}_d)$, $\tau_a = (\tau_0, \tau_i)$, $\tau_0 = I$, $\tau_i\,(i = 1, 2, 3)$ are the Pauli matrices of the flavour group $SU(2)_f$. The constants of the four-quark interactions are G_S for the scalar and pseudoscalar cases, G_V for the vector and the axial-vector cases. The current mass term explicitly breaks the $SU(2)_L \otimes SU(2)_R$ chiral symmetry of the Lagrangian (1). In what follows, we shall only consider the isospin symmetrical case $\widehat{m}_u = \widehat{m}_d = \widehat{m}$. With boson fields introduced in the standard way, the Lagrangian takes the form

$$\begin{aligned} L(q, \bar{\sigma}, \tilde{\pi}, \tilde{v}, \tilde{a}) \;\; &= \;\; \bar{q}\left(i\gamma^\mu\partial_\mu - \widehat{m} + \bar{\sigma} + i\gamma_5\tilde{\pi} + \gamma^\mu\tilde{v}_\mu + \gamma_5\gamma^\mu\tilde{a}_\mu\right)q \\ &- \;\; \frac{\bar{\sigma}_a^2 + \tilde{\pi}_a^2}{2G_S} + \frac{\tilde{v}_{\mu a}^2 + \tilde{a}_{\mu a}^2}{2G_V}. \end{aligned} \tag{2}$$

Here $\bar{\sigma} = \bar{\sigma}_a\tau_a,\ \tilde{\pi} = \tilde{\pi}_a\tau_a,\ \tilde{v}_\mu = \tilde{v}_{\mu a}\tau_a,\ \tilde{a}_\mu = \tilde{a}_{\mu a}\tau_a$. The vacuum expectation value of the scalar field $\bar{\sigma}_0$ turns out to be different from zero ($<\bar{\sigma}_0> \neq 0$). To obtain the physical field $\tilde{\sigma}_0$ with $<\tilde{\sigma}_0> = 0$ one performs a field shift leading to a new quark mass m to be identified with the mass of the constituent quarks

$$\bar{\sigma}_0 - \widehat{m} = \tilde{\sigma}_0 - m, \qquad \bar{\sigma}_i = \tilde{\sigma}_i, \tag{3}$$

where m is determined from the gap equation (see (13) below).

Let us integrate out the quark fields in the generating functional associated with the Lagrangian (2). Evaluating the resulting quark determinant by a loop expansion one obtains

$$\begin{aligned} L(\tilde{\sigma}, \tilde{\pi}, \tilde{v}, \tilde{a}) \;\; &= \;\; -i\mathrm{Tr}\ln\left[1 + (i\gamma^\mu\partial_\mu - m)^{-1}(\tilde{\sigma} + i\gamma_5\tilde{\pi} + \gamma^\mu\tilde{v}_\mu + \gamma_5\gamma^\mu\tilde{a}_\mu)\right]_\Lambda \\ &- \;\; \frac{\bar{\sigma}_a^2 + \tilde{\pi}_a^2}{2G_S} + \frac{\tilde{v}_{\mu a}^2 + \tilde{a}_{\mu a}^2}{2G_V}. \end{aligned} \tag{4}$$

The NJL model belongs to the set of nonrenormalizable theories. Hence, to define it completely as an effective model, a regularization scheme must be specified to deal with the quark-loop integrals in harmony with general symmetry requirements. As a result, an additional parameter Λ appears, which characterizes the scale of the quark-antiquark forces responsible for the dynamic chiral symmetry breaking. From the meson mass spectrum it is known that $\Lambda \sim 1\,\mathrm{GeV}$. Here, we shall make use of the a modified Pauli–Villars [18] regularization, which preserves gauge invariance and chiral symmetry. In this

form it was used in [19]-[20]. The Pauli–Villars cut-off Λ is introduced in the following way,

$$e^{-im^2z} \to R(z) = e^{-im^2z}\left[1 - (1 + iz\Lambda^2)e^{-iz\Lambda^2}\right], \tag{5}$$

$$m^2 e^{-im^2z} \to iR'(z) = m^2 R(z) - iz\Lambda^4 e^{-iz(\Lambda^2+m^2)}, \tag{6}$$

where only one Pauli–Villars regulator has been introduced. In this case the expressions for the basic loop integrals I_i coincide with those obtained by the usual covariant cut-off scheme. Let us give our definitions for these integrals. To simplify the formulae we introduce the following notation

$$\Delta(p) = \frac{1}{p^2 - m^2}, \qquad \tilde{d}^4q = \frac{d^4q}{(2\pi)^4}. \tag{7}$$

Then we have

$$I_1 = iN_c \int \tilde{d}^4q\Delta(q) = \frac{N_c}{(4\pi)^2}\left[\Lambda^2 - m^2\ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right], \tag{8}$$

$$I_2(p^2) = -iN_c \int \tilde{d}^4q\Delta(q)\Delta(q+p)$$

$$= \frac{N_c}{16\pi^2}\int_0^1 dy \int_0^\infty \frac{dz}{z} R(z)e^{\frac{i}{4}zp^2(1-y^2)}, \tag{9}$$

$$J_2(p^2) = \frac{3N_c}{32\pi^2}\int_0^1 dy(1-y^2)\int_0^\infty \frac{dz}{z} R(z)e^{\frac{i}{4}zp^2(1-y^2)}, \tag{10}$$

$$I_3(p_1,p_2) = -iN_c \int \tilde{d}^4q\Delta(q)\Delta(q+p_1)\Delta(q+p_2), \tag{11}$$

$$I_4(p_1,p_2,p_3) = -iN_c \int \tilde{d}^4q\Delta(q)\Delta(q+p_1)\Delta(q+p_2)\Delta(q+p_3). \tag{12}$$

Consider the first terms of the logarithm expansion in (4). From the requirement for the terms linear in $\tilde{\sigma}$ to vanish we get a modified gap equation

$$m - \widehat{m} = 8mG_S I_1. \tag{13}$$

The terms quadratic in the boson fields lead to the amplitudes

$$\Pi^{PP}(p^2) = \left[8I_1 - G_S^{-1} + p^2 g^{-2}(p^2)\right]\varphi_P^+\varphi_P^-, \tag{14}$$

$$\Pi^{SS}(p^2) = \left[8I_1 - G_S^{-1} + (p^2 - 4m^2)g^{-2}(p^2)\right]\varphi_S^+\varphi_S^-, \tag{15}$$

$$\Pi^{VV}(p^2) = \left[g^{\mu\nu}G_V^{-1} + 4(p^\mu p^\nu - g^{\mu\nu}p^2)g_V^{-2}(p^2)\right]\varepsilon_\mu^{*V}(p)\varepsilon_\nu^V(p), \tag{16}$$

$$\Pi^{AA}(p^2) = \Big[g^{\mu\nu}\left(G_V^{-1} + 4m^2 g^{-2}(p^2)\right) + 4(p^\mu p^\nu - g^{\mu\nu}p^2)g_V^{-2}(p^2)\Big]\varepsilon_\mu^{*A}(p)\varepsilon_\nu^A(p), \tag{17}$$

$$\Pi^{PA}(p^2) = 2img^{-2}(p^2)p^\mu\varepsilon_\mu^{*A}(p)\varphi_P^-, \tag{18}$$

$$\Pi^{AP}(p^2) = -2img^{-2}(p^2)p^\mu\varepsilon_\mu^A(p)\varphi_P^+. \tag{19}$$

Here $\varepsilon_\mu^V(p), \varepsilon_\mu^A(p)$ are the polarization vectors of the vector and axial-vector fields. We have introduced the symbols $\varphi_P^- = 1$ and $\varphi_S^- = 1$ to explicitly show the pseudoscalar and scalar field contents of the pertinent two-point functions. The functions $g(p^2)$ and $g_V(p^2)$ are determined by the following integrals

$$g^{-2}(p^2) = 4I_2(p^2), \tag{20}$$

$$g_V^{-2}(p^2) = \frac{2}{3} J_2(p^2). \tag{21}$$

Let us diagonalize the quadratic form (14) + (17) + (18) + (19) by redefining the axial fields

$$\begin{aligned} \varepsilon_\mu^A(p) &\rightarrow \varepsilon_\mu^A(p) - i\beta(p^2) p_\mu \varphi_P^-, & (22) \\ \varepsilon_\mu^{*A}(p) &\rightarrow \varepsilon_\mu^{*A}(p) + i\beta(p^2) p_\mu \varphi_P^+. & (23) \end{aligned}$$

This determines the function $\beta(p^2)$,

$$\beta(p^2) = \frac{8mI_2(p^2)}{G_V^{-1} + 16m^2 I_2(p^2)}. \tag{24}$$

Consequently, one has no more mixing between pseudoscalar and axial-vector fields. The self-energy of the pseudoscalar field takes the form

$$\Pi^{PP}(p^2) = \left[8I_1 - G_S^{-1} + p^2 g^{-2}(p^2)\left(1 - 2m\beta(p^2)\right)\right] \varphi_P^+ \varphi_P^-. \tag{25}$$

Now we can construct special boson variables that will describe the observed mesons. These field functions ϕ^2 correspond to bound quark – antiquark states and are derived via the following transformations

$$\begin{aligned} \tilde{\pi}^a(p) &= Z_\pi^{-1/2} g_\pi(p^2) \pi^a(p), & (26) \\ \tilde{\sigma}^a(p) &= Z_\sigma^{-1/2} g(p^2) \sigma^a(p), & (27) \\ \tilde{v}^a(p) &= \frac{1}{2} Z_v^{-1/2} g_V(p^2) v^a(p), & (28) \\ \tilde{a}^a(p) &= \frac{1}{2} Z_a^{-1/2} g_V(p^2) a^a(p), & (29) \end{aligned}$$

where

$$g_\pi(p^2) = \frac{g(p^2)}{\sqrt{1 - 2m\beta(p^2)}} = g(p^2)\sqrt{1 + 16m^2 G_V I_2(p^2)}. \tag{30}$$

The new bosonic fields have the self-energies

$$\begin{aligned} \Pi_{ab}^{\pi,\sigma}(p^2) &= \delta_{ab} Z_{\pi,\sigma}^{-1} \left[p^2 - m_{\pi,\sigma}^2(p^2)\right], \\ \Pi_{\mu\nu,ab}^{v,a}(p^2) &= \delta_{ab} Z_{v,a}^{-1} \left\{p_\mu p_\nu - g_{\mu\nu}\left[p^2 - m_{v,a}^2(p^2)\right]\right\}. & (31) \end{aligned}$$

[2]Here and in the following we will use the common symbol ϕ for the all set of meson fields: $\pi_a, \sigma_a, v_a, a_a$.

The p^2-dependent masses are equal to

$$
\begin{aligned}
m_\pi^2(p^2) &= (G_S^{-1} - 8I_1) g_\pi^2(p^2) = \widehat{m}(mG_S)^{-1} g_\pi^2(p^2), && (32)\\
m_\sigma^2(p^2) &= \left[1 - 2m\beta(p^2)\right] m_\pi^2(p^2) + 4m^2, && (33)\\
m_v^2(p^2) &= \frac{g_V^2(p^2)}{4G_V} = \frac{3}{8G_V J_2(p^2)}, && (34)\\
m_a^2(p^2) &= m_v^2(p^2) + 6m^2 \frac{I_2(p^2)}{J_2(p^2)}. && (35)
\end{aligned}
$$

These equations coincide with the conditions for appearance of quark – antiquark bound state as deduced in the pure fermion approach from analysis of the Bethe – Salpeter equations.

The constants Z_ϕ are determined by the requirement that the inverse meson field propagators Π^ϕ satisfy the normalization conditions

$$
\begin{aligned}
\Pi^{\pi,\sigma}(p^2) &= p^2 - m_{\pi,\sigma}^2 + \mathcal{O}\left((p^2 - m_{\pi,\sigma}^2)^2\right),\\
\Pi_{\mu\nu}^{v,a}(p^2) &= -g_{\mu\nu}\left[p^2 - m_{v,a}^2 + \mathcal{O}\left((p^2 - m_{v,a}^2)^2\right)\right], && (36)
\end{aligned}
$$

around the physical mass points $p^2 = m_\phi^2$, respectively. The conditions (36) lead to the values

$$
\begin{aligned}
Z_\pi &= 1 + \frac{m_\pi^2[1 - 2m\beta(m_\pi^2)]}{I_2(m_\pi^2)} \frac{\partial I_2(p^2)}{\partial p^2}\Big|_{p^2=m_\pi^2}, && (37)\\
Z_\sigma &= 1 + \frac{m_\sigma^2 - 4m^2}{I_2(m_\sigma^2)} \frac{\partial I_2(p^2)}{\partial p^2}\Big|_{p^2=m_\sigma^2}, && (38)\\
Z_v &= 1 + \frac{m_v^2}{J_2(m_v^2)} \frac{\partial J_2(p^2)}{\partial p^2}\Big|_{p^2=m_v^2}, && (39)\\
Z_a &= 1 + \frac{m_a^2}{J_2(m_a^2)} \frac{\partial J_2(p^2)}{\partial p^2}\Big|_{p^2=m_a^2} - \frac{6m^2}{J_2(m_a^2)} \frac{\partial I_2(p^2)}{\partial p^2}\Big|_{p^2=m_a^2}. && (40)
\end{aligned}
$$

In the following, when omitting an argument of a running coupling constant or a running mass, we always assume that its value is taken on the mass-shell of the corresponding particle. The symbol of this particle will be used for that. For example, on the pion mass-shell $m_\pi^2(p^2 = m_\pi^2) = m_\pi^2$, $\beta(m_\pi^2) = \beta_\pi$ and so on.

Using the expressions (31), one can obtain the two-point meson Green functions $\Delta^\phi(p)$. For example, in the scalar and vector field case the relations

$$
\Pi_{ab}^\sigma(p^2)\Delta_{bc}^\sigma(p^2) = \delta_{ac}, \quad \Pi_{\mu\nu,ab}^v(p^2)\Delta_{bc}^{v,\nu\sigma}(p) = \delta_{ac}\delta_\mu^\sigma \qquad (41)
$$

give

$$
\Delta_{ab}^\sigma(p^2) = \frac{\delta_{ab} Z_\sigma}{p^2 - m_\sigma^2(p^2)}, \quad \Delta_{ab}^{v,\mu\nu}(p) = \frac{\delta_{ab} Z_v}{m_v^2(p^2)} \frac{p^\mu p^\nu - g^{\mu\nu} m_v^2(p^2)}{p^2 - m_v^2(p^2)}. \qquad (42)
$$

This picture corresponds to the calculations in the framework of the pure fermionic NJL model where the Bethe–Salpeter equation sums an infinite class of fermion bubble diagrams.

2.2 The non-linear approach

Phenomenological meson fields with appropriate transformation properties under a non-linear action of the chiral group can be introduced as follows [21], [22]. Let G be a continuous symmetry group of the initial Lagrangian and H a maximum subgroup of group G which leaves the vacuum invariant. Then an arbitrary transformation of the group G can be represented as $G = K(\zeta)H(\eta)$, where ζ, η are the parameters determining the parametrization of the G group space. Acting from the left on the G group element by an arbitrary transformation of the same group $G(g)K(\zeta)H(\eta) = K(\zeta')H(\eta')$, one can find out how the parameters ζ and η are transformed under transformations of the group. It is essential that in this case the transformation for ζ does not involve the parameters η: $\zeta' = \zeta'(\zeta, g)$. Each parameter ζ^i is associated with a local Goldstone field $\pi^i(x)$ so that the local fields $\pi^i(x)$ obey the transformation rule

$$\pi^{i'}(x) = \zeta^{i'}(\pi^i(x), g). \tag{43}$$

The non-linear transformation of the group G on the matter fields is constructed in the following way,

$$Q \to Q' = h(\pi, g)Q, \tag{44}$$

$$R \to R' = h(\pi, g)Rh^\dagger(\pi, g), \tag{45}$$

where Q, are the quark fields, R, the vector, axial-vector or scalar H multiplets, and $h(\pi, g) \in H$.

Let us consider the Lagrangian (2). In this case $G = SU(2)_L \otimes SU(2)_R$. The quark field $q(x)$ can be represented as $q(x) = q_L(x) + q_R(x)$, where $q_L(x) = P_L q(x)$, $q_R(x) = P_R q(x)$. The projection operators $P_{L,R}$ are $P_{R,L} = (1 \pm \gamma_5)/2$. The fields $q_{L,R}(x)$ transform linearly under action of chiral subgroups $SU(2)_{L,R}$:

$$q_L(x) \to g_L(x)q_L(x), \qquad q_R(x) \to g_R(x)q_R(x). \tag{46}$$

Let us introduce the notation $\bar{\sigma}_a + i\gamma_5\tilde{\pi}_a = M_a$. Then we can write $\bar{\sigma} + i\gamma_5\tilde{\pi} = MP_R + M^\dagger P_L$. Now let us represent the complex 2×2 matrix $M = M_a\tau_a$ as a product of the unitary matrix ξ and the Hermitian matrix S

$$M = \xi S \xi. \tag{47}$$

The matrix ξ is parametrized by Goldstone fields, and its transformation law corresponds to the nonlinear transformation (43)

$$\xi \to g_L(x)\xi h^\dagger(\pi, g_{L,R}) = h(\pi, g_{L,R})\xi g_R^\dagger(x). \tag{48}$$

The map $\xi(\pi)\colon G/H \to G$ is thus the local section of the principal H-bundle $G \to G/H$, where $h(\pi, g) \in H$ is a compensating H transformation which brings us back to our canonical choice for coset representative in the new coset specified by π'. The exponential

parametrization $\xi(\pi) = \exp[i\pi/(2F)]$ corresponds to the choice of a normal coordinate system in the coset space G/H. New quark variables

$$Q_R = \xi q_R, \qquad Q_L = \xi^\dagger q_L, \qquad Q = Q_R + Q_L \tag{49}$$

are transformed by the nonlinear representation of the group G, eq.(44), and can be used to describe the constituent quark fields in the approach under consideration. Let us rewrite the Lagrangian (2) as

$$\begin{aligned} L &= \bar{Q}\left[i\gamma^\mu\nabla_\mu + S - \frac{1}{2}(\Sigma + \gamma_5\Delta) + \gamma^\mu\left(W_\mu^{(+)} - \gamma_5 W_\mu^{(-)}\right)\right]Q \\ &- \frac{1}{4G_S}\mathrm{Tr}S^2 + \frac{1}{4G_V}\mathrm{Tr}\left(W_\mu^{(+)}W_\mu^{(+)} + W_\mu^{(-)}W_\mu^{(-)}\right). \end{aligned} \tag{50}$$

Here we use the following notations:

$$\Sigma = \xi^\dagger\widehat{m}\xi^\dagger + \xi\widehat{m}\xi, \qquad \Delta = \xi^\dagger\widehat{m}\xi^\dagger - \xi\widehat{m}\xi, \tag{51}$$

$$\nabla_\mu = \partial_\mu + \Gamma_\mu - \frac{i}{2}\gamma_5\xi_\mu, \tag{52}$$

$$\Gamma_\mu = \frac{1}{2}\left(\xi\partial_\mu\xi^\dagger + \xi^\dagger\partial_\mu\xi\right), \quad \xi_\mu = i(\xi\partial_\mu\xi^\dagger - \xi^\dagger\partial_\mu\xi). \tag{53}$$

To describe the vector $W_\mu^{(+)}$ and axial-vector $W_\mu^{(-)}$ mesons we use new variables

$$W_\mu^{(\pm)} = \frac{1}{2}\left[\xi^\dagger\left(\tilde{v}_\mu + \tilde{a}_\mu\right)\xi \pm \xi\left(\tilde{v}_\mu - \tilde{a}_\mu\right)\xi^\dagger\right]. \tag{54}$$

We take the matrix S in the form $S = \widehat{m} - m + s(x)$. Then the condition for the tadpole not to appear (after integration over quark variables) in the case of the scalar field $s(x)$ will be the familiar gap equation (13).

Among the terms quadratic in meson fields only the amplitudes

$$\Pi^{PP}(p^2) = \frac{4}{F^2}\left[2\widehat{m}(\widehat{m} - m)I_1 + (m - \widehat{m})^2 p^2 I_2(p^2)\right]\varphi^i\varphi^j, \tag{55}$$

$$\Pi^{PA}(p^2) = -\frac{8im}{F}(m - \widehat{m})p^\mu\epsilon_\mu^{j*}(p)I_2(p^2)\varphi^i, \tag{56}$$

$$\Pi^{AP}(p^2) = \frac{8im}{F}(m - \widehat{m})p^\mu\epsilon_\mu^{i}(p)I_2(p^2)\varphi^j \tag{57}$$

will be different than in the linear case (see (14)-(19)). Let us consider a standard replacement in a case like this

$$\epsilon_\mu^i(p) \to \epsilon_\mu^i(p) + ip_\mu\tilde{\beta}\varphi^i, \quad \epsilon_\mu^{j*}(p) \to \epsilon_\mu^{j*}(p) - ip_\mu\tilde{\beta}\varphi^j. \tag{58}$$

The condition for cancellation of nondiagonal terms (absence of the pseudoscalar – axial-vector transition) fixes the form of the function $\tilde{\beta}(p^2)$:

$$\tilde{\beta}(p^2) = \frac{8m(m - \widehat{m})I_2(p^2)}{F[G_V^{-1} + 16m^2 I_2(p^2)]}. \tag{59}$$

The self-energy of the pseudoscalar mode takes the form

$$\Pi^{PP}(p^2) = \frac{4(m-\widehat{m})^2}{F^2}\left[\frac{2\widehat{m}}{\widehat{m}-m}I_1 + p^2 I_2(p^2)\left(1 - \frac{2m\tilde{\beta}(p^2)F}{m-\widehat{m}}\right)\right]\varphi^i\varphi^j. \qquad (60)$$

Thus, in the nonlinear case only the pseudoscalar field $\pi(p)$ will have a renormalization other than in the linear approach. The physical field $\pi^{ph}(p)$ should be introduced by the following replacement

$$\pi(p) = Z_\pi^{-1/2}\tilde{g}_\pi(p^2)\pi^{ph}(p). \qquad (61)$$

For other fields the transformations remain unchanged (see (27)-(29)). The function $\tilde{g}_\pi(p^2)$ is determined from (60)

$$\tilde{g}_\pi^2(p^2) = \frac{F^2[1+16m^2G_VI_2(p^2)]}{4(m-\widehat{m})^2I_2(p^2)} = \left(\frac{Fg_\pi}{m-\widehat{m}}\right)^2. \qquad (62)$$

Hence it follows in particular that all equations for the masses of collective modes coincide with the ones in the linear case (32)-(35). This also applies to the equation of the pion mass, as seen from the chain of transformations

$$m_\pi^2(p^2) = \frac{8\widehat{m}}{F^2}(m-\widehat{m})\tilde{g}_\pi^2(p^2)I_1 = \frac{\widehat{m}[1+16m^2G_VI_2(p^2)]}{4mG_SI_2(p^2)}, \qquad (63)$$

where we employed the gap equation (13). The form factor $f_\pi(p^2)$ appearing in the vertex of the weak pion decay $\pi \to \ell\nu_\ell$ is easily found to be

$$f_\pi(p^2) = \frac{4Z_\pi^{-1/2}m(m-\widehat{m})\tilde{g}_\pi(p^2)I_2(p^2)}{F[1+16m^2G_VI_2(p^2)]} = \frac{m}{\sqrt{Z_\pi}g_\pi(p^2)}. \qquad (64)$$

The latter equality reveals that it coincides with a similar expression derived in the linear model [17]. Since the Bethe–Salpeter equation for the masses of collective states and the expression for the constant f_π coincide in the two approaches under consideration, there exists a unified approach to construct chiral expansions.

2.3 Chiral expansion

At low energies the behaviour of scattering amplitudes or matrix elements for currents can be described in terms of Taylor expansions in powers of momenta. Yet, singularities, arising from the presence of light pseudoscalar particles in the theory, restrict the applicability of Taylor expansions. One has to take into account all these singularities to extend the divergence region for the series in momenta. This is possible because it is quite clear why the pion mass is small. The pion is a Goldstone boson and its mass is expressed in terms of current quarks which make it different from zero. Current quark masses are small and can be taken into account through perturbation theory. New combined Taylor expansion in momenta and current quark masses arising in this case form the basis of chiral perturbation theory [23].

Let $\overset{\circ}{m}$ and f be the values of the constituent quark mass m and the pion decay constant f_π in the chiral limit where $\widehat{m} = 0$. In this case the pion mass is zero. The chiral series for this quantity begins with a term linear in $\widehat{m}$:

$$\overset{\circ}{m}{}^2_\pi = \frac{\widehat{m}\,\overset{\circ}{m}}{G_S f^2}. \tag{65}$$

The weak pion decay constant f_π is (see eq.(64))

$$f_\pi = \frac{m}{\sqrt{Z_\pi} g_\pi}. \tag{66}$$

Using (30) and (20), we arrive at

$$f_\pi^2 = 4 Z_\pi^{-1} \delta m^2 I_2(m_\pi^2). \tag{67}$$

Hence, at $\widehat{m} \to 0$ we get

$$f^2 = 4 \overset{\circ}{\delta} \overset{\circ}{m}{}^2 \overset{\circ}{I}_2, \tag{68}$$

which is non–vanishing in the chiral limit. Here and below we use the following notation

$$I_2 = I_2(0), \quad \overset{\circ}{I}_2 = \lim_{\widehat{m}\to 0} I_2. \tag{69}$$

$$\delta = 1 - 2m\beta_\pi, \quad \overset{\circ}{\delta} = \lim_{\widehat{m}\to 0} \delta. \tag{70}$$

All these quantities are convenient to simplify the formulae. Note that the equality

$$1 - 4G_V f^2 = \overset{\circ}{\delta} \tag{71}$$

is valid. We shall employ the gap equation (13) to expand the constituent quark mass in a series in powers of the current quark mass,[3]

$$m = \sum_{i=0}^{\infty} c_i^{(m)} \widehat{m}^i. \tag{72}$$

Here

$$c_0^{(m)} = \overset{\circ}{m} = \lim_{\widehat{m}\to 0} m, \qquad c_i^{(m)} = \lim_{\widehat{m}\to 0} \frac{1}{i!} \frac{\partial^i m}{\partial \widehat{m}^i}. \tag{73}$$

The mass $\overset{\circ}{m}$ is a solution of the gap equation at $\widehat{m} = 0$. To find other coefficients of the series we differentiate equation (13) with respect to the current quark mass $\widehat{m}$. As a result, we get the following expression

$$4G_S M^2(m) = \frac{1}{m'} - \frac{\widehat{m}}{m}. \tag{74}$$

[3]We remark that only analytic terms in the current quark masses can appear within the Hartree approximation employed here.

We introduced the abbreviation $M^2(m) = 4m^2 I_2$. In the case of exact chiral symmetry with $\widehat{m} = 0$ one can derive the value of $M^2(\overset{\circ}{m})$ from equation (68). Bearing in mind these remarks, we get the following relation from the above equation:

$$c_1^{(m)} = \frac{\overset{\circ}{\delta}}{4G_S f^2}. \tag{75}$$

Other coefficients of the series are calculated through successive differentiation of the gap equation. For example, differentiating equation (74) we get

$$8G_S M M' = \frac{\widehat{m}m'}{m^2} - \frac{1}{m} - \frac{m''}{(m')^2}. \tag{76}$$

On the other hand, relying on the definition of M one has

$$\frac{MM'}{mm'} = \frac{M^2}{m^2} - \frac{3h_1}{4\pi^2}, \qquad h_1 = \left(\frac{\Lambda^2}{\Lambda^2 + m^2}\right)^2. \tag{77}$$

In the chiral limit it follows from these two equations that

$$c_2^{(m)} = \frac{3(c_1^{(m)})^2}{2\overset{\circ}{m}} \left(\frac{\overset{\circ}{m}{}^2 \overset{\circ}{h}_1 \overset{\circ}{\delta}}{2\pi^2 f^2} - 1\right). \tag{78}$$

For simplicity we use the symbol $\overset{\circ}{h}_1$ to denote the chiral limit

$$\overset{\circ}{h}_1 = \lim_{\widehat{m}\to 0} h_1 = \left(\frac{\Lambda^2}{\Lambda^2 + \overset{\circ}{m}{}^2}\right)^2. \tag{79}$$

Thus we obtain

$$m = \overset{\circ}{m}\left[1 + \frac{\overset{\circ}{m}_\pi{}^2 \overset{\circ}{\delta}}{4\overset{\circ}{m}{}^2} - \frac{3\overset{\circ}{m}_\pi{}^4 \overset{\circ}{\delta}{}^2}{32\overset{\circ}{m}{}^4}\left(1 - \frac{\overset{\circ}{m}{}^2 \overset{\circ}{h}_1 \overset{\circ}{\delta}}{2\pi^2 f^2}\right) + \mathcal{O}(\overset{\circ}{m}{}^6_\pi)\right]. \tag{80}$$

Another important example is the chiral expansion for the pion mass. While in the case of the quark mass we employed the gap equation, here we need a pion mass equation (32), which can be conveniently represented as

$$\left(m_\pi^2 - 4m\widehat{m}\frac{G_V}{G_S}\right) I_2(m_\pi^2) = \frac{\widehat{m}}{4mG_S}. \tag{81}$$

It follows from the equation that this kind of expansion begins with a term proportional to $\widehat{m}$. We have already called it $\overset{\circ}{m}{}^2_\pi$ (see (65)). Let us find the first few coefficients of the series

$$m_\pi^2 = \sum_{i=1}^{\infty} c_i^{(m_\pi)} \widehat{m}^i. \tag{82}$$

Obviously, $c_1^{(m_\pi)} = \overset{\circ}{m}(G_S f^2)^{-1}$. To find other coefficients of the series we represent equation (81) as

$$4mG_S\left[\sum_{i=0}^{\infty} c_{i+1}^{(m_\pi)}\widehat{m}^i - 4m\frac{G_V}{G_S}\right] I_2(m_\pi^2) = 1. \tag{83}$$

Using the expansion (80) we isolate combinations to the same powers of $\widehat{m}$ on the left-hand side of the equation and set them equal to zero. Thus we can calculate coefficients of the expansion (82). For example

$$c_2^{(m_\pi)} = \frac{\overset{\circ}{\delta}}{(2f^2G_S)^2}\left[\frac{\overset{\circ}{m}{}^2\overset{\circ}{h}_1}{2\pi^2 f^2}\,\overset{\circ}{\delta}\,(3\,\overset{\circ}{\delta}\,-1) + (1-2\,\overset{\circ}{\delta})\right]. \tag{84}$$

$$c_3^{(m_\pi)} = \frac{\overset{\circ}{\delta}{}^2}{4\overset{\circ}{m}(2f^2G_S)^3}\left\{8\overset{\circ}{\delta}-3+\frac{\overset{\circ}{m}{}^2\overset{\circ}{h}_1}{2\pi^2f^2}\left[\frac{\overset{\circ}{\delta}(2-10\overset{\circ}{\delta}+21\overset{\circ}{\delta}{}^2)\overset{\circ}{m}{}^2\overset{\circ}{h}_1}{2\pi^2f^2}\right.\right.$$
$$\left.\left.+7\overset{\circ}{\delta}-20\overset{\circ}{\delta}{}^2-\frac{4}{5}-\frac{4(15\overset{\circ}{\delta}{}^2-10\overset{\circ}{\delta}+2)\overset{\circ}{m}{}^2}{5(\Lambda^2+\overset{\circ}{m}{}^2)}\right]\right\}. \tag{85}$$

Below we give the results of similar calculations for the pion decay constant f_π on the basis of the expression (66).

$$f_\pi = \sum_{i=0}^{\infty} c_i^{(f_\pi)}\widehat{m}^i. \tag{86}$$

In this case the corresponding coefficients of the chiral series are

$$c_0^{(f_\pi)} = f, \tag{87}$$

$$c_1^{(f_\pi)} = \frac{\overset{\circ}{\delta}{}^2}{4\overset{\circ}{m}fG_S}\left[1-\frac{3\,\overset{\circ}{m}{}^2\overset{\circ}{h}_1\overset{\circ}{\delta}}{(2\pi f)^2}\right], \tag{88}$$

$$c_2^{(f_\pi)} = -\frac{3\overset{\circ}{\delta}{}^3(2-\overset{\circ}{\delta})}{2f(4\overset{\circ}{m}fG_S)^2} + \frac{\overset{\circ}{\delta}{}^2\overset{\circ}{h}_1}{2f(4\pi f^2G_S)^2}\left\{3\overset{\circ}{\delta}{}^2\left[\frac{\Lambda^2+2\overset{\circ}{m}{}^2}{\Lambda^2+\overset{\circ}{m}{}^2}\right.\right.$$
$$\left.\left.+\frac{3}{2}(1-\overset{\circ}{\delta})-\frac{9\overset{\circ}{m}{}^2\overset{\circ}{h}_1\overset{\circ}{\delta}}{(4\pi f)^2}(2-\overset{\circ}{\delta})\right]+\frac{\overset{\circ}{m}{}^2\overset{\circ}{h}_1\overset{\circ}{\delta}}{(2\pi f)^2}-\frac{\Lambda^2+3\overset{\circ}{m}{}^2}{5(\Lambda^2+\overset{\circ}{m}{}^2)}\right\}. \tag{89}$$

Concluding the section we point out that chiral expansions derived here for the main meson characteristics prove to be helpful in establishing correspondence between the results obtained here and the known low-energy theorems of current algebra. We shall use them to analyze $\pi\pi$ scattering and to consider the scalar radius of the pion.

3 $\pi\pi$-scattering

The formal scheme described in the previous section gives the possibility of evaluating any mesonic N-point function through the parameters of the model: $\Lambda, \widehat{m}, G_S, G_V$. Let us use it for the evaluation of the $\pi\pi$-scattering amplitude. We have done that in the same approach earlier [2], but without considering the spin-one mesons. Now we would like to explore the role of vector and axial-vector particles in this process.

The model with linearly realized chiral symmetry was already employed for this purpose [15]. However, the results of [15] should be considered only as a first approximation to this problem. The $\pi - a_1$ mixing was neglected there. This breaks chiral symmetry, which is recovered only if the constituent quark mass m goes to infinity. In our case we exactly reproduce Weinberg's result at the level E^2 and we also evaluate all higher order corrections in E^2.

The extended NJL model with nonlinearly realized chiral symmetry has not been used to study low-energy $\pi\pi$ scattering so far, though some general conclusions can be drawn from the results of [14]. Chiral expansion parameters L_i were calculated in the model, the standard heat kernel method being used to get the first terms of the effective meson Lagrangian (up-to-and-including the $\mathcal{O}(E^4)$ terms). We shall advance farther and find a general (to any accuracy in E^2) form of the $\pi\pi$ scattering amplitude in the principal approximation in $1/N_c$.

In our calculations we use the conventional Mandelstam variables:

$$s = (q_1 + q_2)^2, \quad t = (q_1 - q_3)^2, \quad u = (q_1 - q_4)^2 \tag{90}$$

for the scattering process $\pi^a(q_1) + \pi^b(q_2) \to \pi^c(q_3) + \pi^d(q_4)$. The $\pi\pi$ scattering amplitude $T_{ab;cd}$ has the following isotopic structure

$$T_{ab;cd}(s,t,u) = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,s,u) + \delta_{ad}\delta_{cb}A(u,t,s). \tag{91}$$

It follows that the amplitudes with definite isospin are

$$\begin{aligned} T^0(s,t,u) &= A(t,s,u) + A(u,t,s) + 3A(s,t,u), \\ T^1(s,t,u) &= A(t,s,u) - A(u,t,s), \\ T^2(s,t,u) &= A(t,s,u) + A(u,t,s). \end{aligned} \tag{92}$$

Meson tree diagrams corresponding to the amplitude $A(s,t,u)$ are of the same form both in the linear and nonlinear approach (see Fig.1). Only the internal structure of quark-loop-based meson vertices will be different.

3.1 The amplitude (linear approach)

Let us calculate the amplitude $A(s,t,u)$ in the linear approach. The vector meson and scalar meson exchange diagrams of Fig.1 include the triangular vertices $\rho \to \pi\pi$ and

$\sigma \to \pi\pi$. We calculate these vertices in accordance with the diagrams of Fig.2, taking into account the πa_1 mixing. The $\rho^a_\mu(p) \to \pi^b(p_1)\pi^c(p_2)$ amplitude is equal to

$$M_{\rho\pi\pi} = \frac{1}{4}\mathrm{Tr}(\tau_a[\tau_b,\tau_c]_-)(p_1-p_2)^\mu \varepsilon_\mu(p) f_{\rho\pi\pi}(p_1,p_2), \tag{93}$$

where

$$f_{\rho\pi\pi}(p_1,p_2) = \frac{g_\rho(p^2)}{\sqrt{Z_\rho}} F(p_1,p_2). \tag{94}$$

The function $F(p_1,p_2)$ has the form (for on-shell pions)

$$F(p^2) = \frac{1}{Z_\pi}\Big\{1 - \frac{m\beta_\pi p^2}{m_\rho^2(p^2)} + \frac{\delta}{p^2-4m_\pi^2}\Big[(p^2-2m_\pi^2)\left(\frac{I_2(p^2)}{I_2(m_\pi^2)}-1\right) + 2m_\pi^4 \frac{I_3(-p_1,p_2)}{I_2(m_\pi^2)}\Big]\Big\}. \tag{95}$$

Here $p = p_1+p_2$, with p_1, p_2 being the pion momenta. This function obeys the requirement of universality of electromagnetic interactions, $F(0) = 1$. To see this one has to use the equality

$$I_2(0) - I_2(m_\pi^2) - m_\pi^2 I_3(-p_1,p_2)\big|_{p^2=0} = 2m_\pi^2 \frac{\partial I_2(p^2)}{\partial p^2}\Big|_{p^2=m_\pi^2}. \tag{96}$$

The $\sigma^a(p) \to \pi^b(p_1)\pi^c(p_2)$ amplitude has the form

$$M_{\sigma\pi\pi}(p_1,p_2) = \frac{1}{4}\mathrm{Tr}\left(\tau_a[\tau_b,\tau_c]_+\right) f_{\sigma\pi\pi}(p_1,p_2). \tag{97}$$

The vertex function $f_{\sigma\pi\pi}(p_1,p_2)$ on the pion mass shell is

$$f_{\sigma\pi\pi}(p^2) = \frac{16 m g_\pi^2 g(p^2)}{Z_\pi\sqrt{Z_\sigma}}\left\{\left[1 - \frac{p^2}{4m^2}(1-\delta^2)\right] I_2(p^2) + \frac{m_\pi^2\delta}{2m^2}(1-\delta) I_2(m_\pi^2) + \frac{\delta^2}{2}(p^2-2m_\pi^2) I_3(-p_1,p_2)\right\}. \tag{98}$$

Using these expressions we obtain the contributions of the meson-exchange diagrams in Fig.1 to the amplitude $A(s,t,u)$,

$$A_\rho(s,t,u) = 4G_V\left[\frac{(s-u)F^2(t)}{1-\frac{8}{3}G_V t J_2(t)} + \frac{(s-t)F^2(u)}{1-\frac{8}{3}G_V u J_2(u)}\right]. \tag{99}$$

$$A_\sigma(s,t,u) = \frac{Z_\sigma f^2_{\sigma\pi\pi}(s)}{m_\sigma^2(s)-s}. \tag{100}$$

Now let us consider the remaining diagrams, the boxes. Their total number is 16. In Fig. 3 we display only structurally different diagrams. Their calculation is quite

cumbersome. Here we collect only the final result,

$$A_{rest}(s,t,u) = 4g_\pi^4 Z_\pi^{-2}\Big\{4[s\beta_\pi^2 I_2(m_\pi^2) - I_2(s)] + (1-\delta^2)^2\frac{s}{m^2}$$
$$\times[I_2(s) - I_2(m_\pi^2)] + \frac{2}{3}\beta_\pi^4[(s-u)tJ_2(t) + (s-t)uJ_2(u)]$$
$$- \delta^2[4(s-2m_\pi^2)I_3(q_1,-q_2) - 4\beta_\pi^2 C(s,t,u) + \delta^2 B(s,t,u)]\Big\}, \qquad (101)$$

where

$$B(s,t,u) = (2m_\pi^4 - us)I_4(q_1, q_1+q_2, q_4) + (2m_\pi^4 - ts)I_4(q_1, q_1+q_2, q_3)$$
$$-(2m_\pi^4 - ut)I_4(q_1, q_1-q_3, q_4)]. \qquad (102)$$

$$C(s,t,u) = \left(\frac{s-u}{s+u}\right)\Big\{(t-2m_\pi^2)[I_2(t) - I_2(m_\pi^2)] + 2m_\pi^4 I_3(q_1,q_3)\Big\} + (t \leftrightarrow u). \qquad (103)$$

Let us consider the low-energy expansion of the total $\pi\pi$-scattering amplitude

$$A(s,t,u) = A_\rho(s,t,u) + A_\sigma(s,t,u) + A_{rest}(s,t,u). \qquad (104)$$

The self-consistent procedure to do that is chiral perturbation theory [23]. The chiral expansion in powers of external momenta and current quark masses in the case of the $\pi\pi$ scattering amplitude has to lead to Weinberg's celebrated theorem [16] at the E^2 level,

$$A(s,t,u) = \frac{s - \overset{\circ}{m}{}_\pi^2}{f^2} + \mathcal{O}(E^4), \qquad (105)$$

where $\mathcal{O}(E^4)$ is the short form for the terms of order q^4, $\overset{\circ}{m}{}_\pi^2 q^2$, $\overset{\circ}{m}{}_\pi^4$ and higher. In the case at hand the simplest way to get this result is to consider first of all the momentum expansion and only partly use the quark mass expansion. We shall do a chiral expansion for the pion mass, m_π, and the constituent quark mass, m, only at the last stage. In accordance with these remarks one obtains

$$A_\rho = \frac{4g_\pi^4}{Z_\pi^2}\left(\frac{3s-4m_\pi^2}{m^2}\right)\delta(1-\delta)I_2 + \mathcal{O}(E^4). \qquad (106)$$

$$A_\sigma = \frac{4g_\pi^4}{Z_\pi^2}\Big\{4I_2 + \frac{I_2}{m^2}\left[s\left(2\delta^2-1\right) - m_\pi^2\delta(4\delta-3)\right]$$
$$+\frac{h_1}{8\pi^2 m^2}\left[s - 3\delta^2(s-2m_\pi^2)\right]\Big\} + \mathcal{O}(E^4). \qquad (107)$$

$$A_{rest} = -\frac{4g_\pi^4}{Z_\pi^2}\left\{4I_2(1-\beta_\pi^2 s) + \frac{h_1}{8\pi^2 m^2}\left[s - 3\delta^2(s-2m_\pi^2)\right]\right\} + \mathcal{O}(E^4). \qquad (108)$$

Some remarks are useful here. We use momentum expansions of integrals $I_2(s) = I_2 + (sh_1)/(32\pi^2 m^2) + \ldots$, $I_3(q_1,q_2) = -3h_1/(32\pi^2 m^2) + \ldots$. One can see that not only

divergent terms ($\sim I_2$) contribute at this level. There are finite contributions which are proportional to h_1. Full cancellation of these terms in the full amplitude

$$A(s,t,u) = \frac{4g_\pi^4 \delta I_2}{Z_\pi^2 m^2}(s - m_\pi^2) + \mathcal{O}(E^4) \tag{109}$$

is a good test of self-consistency of our approach. In order to preserve chiral symmetry they must start only at the next-to-leading order. Noting that

$$\frac{4g_\pi^4 \delta I_2}{Z_\pi^2 m^2} = \frac{I_2}{Z_\pi^3 f_\pi^2 I_2(m_\pi^2)} = \frac{1}{f^2} + \mathcal{O}(E^2), \tag{110}$$

we have full agreement of our amplitude with the low-energy theorem (105).

Now let us consider the E^4 correction to Weinberg's result. After some lengthy calculations one obtains

$$A^{(4)}(s,t,u) = \frac{1}{96\pi^2 f^4}\left\{2\bar{l}_1(s - 2\mathring{m}_\pi^2)^2 + \bar{l}_2\left[s^2 + (t-u)^2\right]\right\}. \tag{111}$$

Here

$$\bar{l}_1 = \frac{12\pi^2 f^2}{\mathring{m}^2}\,\mathring{\delta}^2\left[\mathring{\delta} - \frac{\mathring{m}^2 \mathring{h}_1}{\pi^2 f^2} + \frac{9\mathring{\delta}^3 \mathring{m}^4 \mathring{h}_1}{(2\pi f)^4}\right] - 9\mathring{\delta}^4 \mathring{h}_2 - \frac{4\pi^2 f^2}{\mathring{m}^2 \mathring{\delta}}(1-\mathring{\delta}^2)^2, \tag{112}$$

$$\bar{l}_2 = 3\mathring{\delta}^4 \mathring{h}_2 + \frac{2\pi^2 f^2}{\mathring{m}^2 \mathring{\delta}}(1-\mathring{\delta}^2)^2 + 6\mathring{\delta}^2(1-\mathring{\delta}^2)\,\mathring{h}_1, \tag{113}$$

and

$$\mathring{h}_2 = \left(\frac{\Lambda^2 + 3\,\mathring{m}^2}{\Lambda^2 + \mathring{m}^2}\right)\mathring{h}_1\,. \tag{114}$$

Let us compare our result (111) with the known estimations of the $\mathcal{O}(E^4)$ terms. One can see that the structure of the term $A^{(4)}(s,t,u)$ fully agree with the early result of [24]. There parameters $\bar{l}_1$ and $\bar{l}_2$ in the limiting case, when $G_V \to 0$, i.e. $\mathring{\delta} \to 1$, go over into the known result [20], [2]. In the case of $SU(3)$ symmetry this set of parameters are known as L_i. They have been obtained in the paper [14] on the basis of $SU(3) \otimes SU(3)$ NJL Lagrangian with spin-one mesons. The constants L_i specify the general effective meson Lagrangian at order of E^4. One should, however, not compare them directly to those analyzed by Gasser and Leutwyler since their analysis includes the effect of meson loops[4] If one considers the $SU(2)$ limit of the effective meson Lagrangian from [14] and compare it with the known $SU(2)$ effective meson Lagrangian from [20] one can relate the $SU(3)$ low-energy parameters L_i with the $SU(2)$ constants $\bar{l}_i$. In particular we have

$$L_2 = 2L_1 = \frac{\bar{l}_2}{192\pi^2}, \quad L_3 = \frac{\bar{l}_1 - \bar{l}_2}{192\pi^2}. \tag{115}$$

[4] And thus leads to a scale-dependence not present in the NJL model.

Or

$$L_2 = 2L_1 = \frac{1}{64\pi^2}\left[\overset{\circ}{\delta}{}^4\overset{\circ}{h}_2 + \frac{2\pi^2 f^2}{3\overset{\circ}{\delta}\overset{\circ}{m}{}^2}(1-\overset{\circ}{\delta}{}^2)^2 + 2\overset{\circ}{\delta}{}^2(1-\overset{\circ}{\delta}{}^2)\overset{\circ}{h}_1\right]. \tag{116}$$

$$L_3 = \frac{1}{64\pi^2}\left[-4\overset{\circ}{\delta}{}^4\overset{\circ}{h}_2 - \frac{2\pi^2 f^2}{\overset{\circ}{\delta}\overset{\circ}{m}{}^2}(1-\overset{\circ}{\delta}{}^2)^2 + 2\overset{\circ}{\delta}{}^2(4\overset{\circ}{\delta}{}^2-3)\overset{\circ}{h}_1 + \frac{4\pi^2 f^2\overset{\circ}{\delta}{}^3}{\overset{\circ}{m}{}^2}\left(1-\frac{3\overset{\circ}{\delta}\overset{\circ}{m}{}^2\overset{\circ}{h}_1}{4\pi^2 f^2}\right)^2\right]. \tag{117}$$

To compare our formulae with similar expressions derived in [14], we point out the following rules of correspondence:

$$\Gamma(0,x) \sim \frac{4\pi^2 f^2}{3\overset{\circ}{\delta}\overset{\circ}{m}{}^2}, \quad \Gamma(1,x) \sim \overset{\circ}{h}_1, \quad \Gamma(2,x) \sim \overset{\circ}{h}_2, \quad g_A \sim \overset{\circ}{\delta}. \tag{118}$$

Here, we give the notation of [14] on the left side and our equivalents on the right side. Now it is easy to make sure that this part of our calculations yields results which agree fully with earlier estimates.

Let us calculate scattering lengths a_l^I and effective range parameters b_l^I up to and including $\mathcal{O}(E^4)$ terms. The result is given in the Appendix since it is not new. The subsequent numerical estimates will reveal that the E^4 approximation of the extended NJL model is a very good approximation to the Hartree solution.

3.2 The amplitude (non-linear approach)

Let us calculate the $\pi\pi$ scattering amplitude in the nonlinear approach. Though we used the same initial Lagrangian, the scheme developed differs from the approach with linearly realized chiral symmetry. The difference arises from variation in the structure of vertices describing the interaction of Goldstone particles with quarks. Note that the structure of similar vertices for scalar, vector, and axial-vector fields remains unchanged. As before, we begin with the amplitude $\rho_\mu^a(p) \to \pi^b(p_1)\pi^c(p_2)$ (see (93)). To distinguish linear results from their nonlinear counterparts, we shall use symbols with a tilde above them. For example, after calculation of the diagrams in Fig. 4, instead of the function $F(p^2)$ (see (95)) we get a new expression $\tilde{F}(p^2)$:

$$\tilde{F}(p^2) = \frac{4g_\pi^2}{Z_\pi(m-\widehat{m})^2}\left\{\frac{p^2}{12}(1-g_A^2)J_2(p^2) - \widehat{m}(mg_A-\widehat{m})I_2(m_\pi^2) + \frac{(mg_A-\widehat{m})^2}{p^2-4m_\pi^2}\left[(p^2-2m_\pi^2)\left(I_2(p^2)-I_2(m_\pi^2)\right) + 2m_\pi^4 I_3(-p_1,p_2)\right]\right\}. \tag{119}$$

The constant g_A is equal to $1-2\tilde{\beta}_\pi F$. It can be related to the known constant δ: $(mg_A - \widehat{m}) = (m-\widehat{m})\delta$. If $\widehat{m}, p^2 \to 0$, then $\tilde{F}(p^2) \to 0$. This off-shell behaviour of the form

factor essentially differs from the behaviour of $F(p^2)$, where $F(0) = 1$. The contribution to the $\pi\pi$ scattering amplitude from the ρ meson exchange diagram will be described by formula (99), where one should make a change $F(p^2) \to \tilde{F}(p^2)$. We call this contribution $\tilde{A}_\rho(s,t,u)$. For the given amplitude, owing to the above property of the function $\tilde{F}(p^2)$, the chiral expansion begins only with the terms $\mathcal{O}(E^6)$.

The vertex function for the two-pion decay of a scalar particle[5] will also differ from the linear one. It is described by the second group of diagrams in Fig. 4.

$$\begin{aligned}\tilde{f}_{\sigma\pi\pi}(p^2) = \frac{4g(p^2)g_\pi^2}{Z_\pi\sqrt{Z_\sigma}(m-\widehat{m})^2}\Big\{&(1-2g_A)(mg_A-\widehat{m})m_\pi^2 I_2(m_\pi^2)\\ &+\left[mg_A^2+\widehat{m}(1-2g_A)\right]p^2 I_2(p^2) - 4m\widehat{m}(m-\widehat{m})I_2(p^2)\\ &+2m(mg_A-\widehat{m})^2(p^2-2m_\pi^2)I_3(-p_1,p_2)\Big\}. \end{aligned} \tag{120}$$

The form factor $\tilde{f}_{\sigma\pi\pi}(p^2)$ also tends to zero at $\widehat{m}, p^2 \to 0$. Changing $f_{\sigma\pi\pi}$ for $\tilde{f}_{\sigma\pi\pi}$ in (100) we get the contribution to the $\pi\pi$ scattering amplitude from the exchange of the given scalar. We denote it by $\tilde{A}_\sigma(s,t,u)$. The chiral expansion of $\tilde{A}_\sigma(s,t,u)$ begins with terms of order $\mathcal{O}(E^4)$.

Now let us consider the remaining group of diagrams displayed in Fig. 5. Here we do not show diagrams resulting from elimination of πa_1 mixing. These diagrams are easily derived from the given ones by the corresponding inserts in vertices ξ with one pion at the end. Without going into detail, we shall give the total contribution to the $\pi\pi$ scattering amplitude from all the diagrams taken together,

$$\begin{aligned}\tilde{A}_{rest}(s,t,u) = \frac{4g_\pi^4}{Z_\pi^2(m-\widehat{m})^4}\Big\{&(mg_A-\widehat{m})\left[(m-3\widehat{m})(s-m_\pi^2)\right.\\ &\left.+2\widehat{m}(1-2g_A)m_\pi^2\right]I_2(m_\pi^2) + \frac{(1-g_A^2)^2}{24}\left[(s-u)tJ_2(t)+(s-t)uJ_2(u)\right]\\ &+\widehat{m}\left[\widehat{m}(1-g_A)^2 s + 2g_A(mg_A-\widehat{m})s - 4\widehat{m}(m-\widehat{m})^2\right]I_2(s)\\ &+g_A^2(mg_A-\widehat{m})(mg_A-3\widehat{m})s\left[I_2(s)-I_2(m_\pi^2)\right]\\ &+(mg_A-\widehat{m})^2\left[4\widehat{m}(m-\widehat{m})(s-2m_\pi^2)I_3(q_1,-q_2)\right.\\ &\left.-(1-g_A^2)C(s,t,u)-(mg_A-\widehat{m})^2 B(s,t,u)\right]\Big\}. \end{aligned} \tag{121}$$

Here we use the notation (102) and (103) adopted earlier.

The expressions for the separate amplitudes $A_{\rho,\sigma,rest}(s,t,u)$ in the linear and nonlinear approaches are different. Before we will consider the total amplitude $A(s,t,u) = \tilde{A}_\rho + \tilde{A}_\sigma + \tilde{A}_{rest}$ let us show that its chiral expansion coincide up to and including the terms $\mathcal{O}(E^4)$ with the corresponding result in the linear approach. Indeed, in the main $\mathcal{O}(E^2)$

[5]Similarly to the linear case, we shall use the symbol σ for this particle, though in the Lagrangian the symbol s was used.

approximation we get Weinberg's result which follows from the term

$$\frac{4g_\pi^4}{Z_\pi^2(m-\widehat{m})^4}(mg_A-\widehat{m})(m-3\widehat{m})(s-m_\pi^2)I_2(m_\pi^2)=\frac{s-\overset{\circ}{m}{}_\pi^2}{f^2}+\mathrm{O}(E^4). \tag{122}$$

In the $\mathcal{O}(E^4)$ approximation the contribution comes from only two amplitudes, $\tilde{A}_\sigma(s,t,u)$ and $\tilde{A}_{rest}(s,t,u)$. As was established, ρ meson exchange diagrams lead to an amplitude beginning with the term of the order $\mathcal{O}(E^6)$. Here we get

$$\begin{aligned}\tilde{A}_\sigma^{(4)}=&\frac{8\overset{\circ}{m}\widehat{m}}{f^4}\left\{\left[2\overset{\circ}{m}\widehat{m}-s\overset{\circ}{\delta}{}^2-\overset{\circ}{\delta}\,(1-2\overset{\circ}{\delta})\overset{\circ}{m}{}_\pi^2\right]\overset{\circ}{I}_2+\frac{3\overset{\circ}{\delta}{}^2\overset{\circ}{h}_1}{16\pi^2}(s-2\overset{\circ}{m}{}_\pi^2)\right\}\\&+\frac{\overset{\circ}{\delta}}{4\overset{\circ}{m}{}^2f^2}\left[\overset{\circ}{\delta}(s-2\overset{\circ}{m}{}_\pi^2)\left(1-\frac{3\overset{\circ}{m}{}^2\overset{\circ}{\delta}\overset{\circ}{h}_1}{4\pi^2f^2}\right)+\overset{\circ}{m}{}_\pi^2\right]^2,\end{aligned} \tag{123}$$

$$\begin{aligned}\tilde{A}_{rest}^{(4)}=&-\frac{8\overset{\circ}{m}\widehat{m}}{f^4}\left\{\left[2\overset{\circ}{m}\widehat{m}-s\overset{\circ}{\delta}{}^2-\overset{\circ}{\delta}\,(1-2\overset{\circ}{\delta})\overset{\circ}{m}{}_\pi^2\right]\overset{\circ}{I}_2+\frac{3\overset{\circ}{\delta}{}^2\overset{\circ}{h}_1}{16\pi^2}(s-2\overset{\circ}{m}{}_\pi^2)\right\}\\&+\frac{(1-\overset{\circ}{\delta}{}^2)^2}{6f^4}[(s-u)t+(s-t)u]\,\overset{\circ}{I}_2-\frac{\overset{\circ}{m}{}_\pi^2\overset{\circ}{\delta}}{4\overset{\circ}{m}{}^2f^2}\left[2s\overset{\circ}{\delta}+(1-4\overset{\circ}{\delta})\overset{\circ}{m}{}_\pi^2\right]\\&+\frac{\overset{\circ}{\delta}{}^2\overset{\circ}{h}_1}{8\pi^2f^4}\left\{s(s-\overset{\circ}{m}{}_\pi^2)\,\overset{\circ}{\delta}{}^2-\overset{\circ}{m}{}_\pi^2\left[3s(1-\overset{\circ}{\delta})+2\overset{\circ}{m}{}_\pi^2\,(3\overset{\circ}{\delta}-2)\right]\right.\\&\qquad\left.+(1-\overset{\circ}{\delta}{}^2)[(s-u)(t+\overset{\circ}{m}{}_\pi^2)+(s-t)(u+\overset{\circ}{m}{}_\pi^2)]\right\}\\&-\frac{\overset{\circ}{\delta}{}^4\overset{\circ}{h}_2}{8\pi^2f^4}\left[2\overset{\circ}{m}{}_\pi^4+ut-s(t+u)\right].\end{aligned} \tag{124}$$

Summing the expressions, we arrive at a result coinciding excately with the result of similar linear calculations (111). Noteworthy is that contributions proportional to $\widehat{m}$ are fully cancelled by the summation. Thus, despite all the differences in the two approaches, they are equivalent up-to-and-including the terms of the order $\mathcal{O}(E^4)$ in the chiral expansion.

3.3 Equivalence of the linear and the non-linear approach

Haag's theorem [25] in axiomatic field theory states the independence of the S-matrix elements on mass-shell from the choice of interpolating fields. In the framework of the Lagrangian approach the same result exists [26]. In our case it means in particular that if we did all correctly the total amplitude $A(s,t,u)$ should be the same in both cases. To demonstrate it let us first consider the $\rho\pi\pi$ form factors $F(p^2)$ and $\tilde{F}(p^2)$ (see (95) and (119)). One then obtains

$$\tilde{F}(p^2)-F(p^2)=\frac{1}{Z_\pi}\left(\frac{m}{m-\widehat{m}}\right)\left[\frac{p^2}{m_\rho^2(p^2)}-1\right]. \tag{125}$$

It is clear now that on the ρ mass shell these form factors coincide with each other. We have a similar behaviour for $f_{\sigma\pi\pi}$ and $\tilde{f}_{\sigma\pi\pi}$ (see expressions (98) and (120)),

$$\tilde{f}_{\sigma\pi\pi}(p^2) - f_{\sigma\pi\pi}(p^2) = \frac{4g_\pi^2 g(p^2) I_2(p^2)}{Z_\pi \sqrt{Z_\sigma}(m-\widehat{m})} \left[p^2 - m_\sigma^2(p^2)\right]. \tag{126}$$

Again at $p^2 = m_\sigma^2$ one gets zero for this difference.

We are ready now to compare the amplitudes

$$\Delta_\alpha(s,t,u) = \tilde{A}_\alpha(s,t,u) - A_\alpha(s,t,u), \tag{127}$$

where $\alpha = \rho, \sigma, rest$. The result can be written as

$$\begin{aligned}\Delta_\rho(s,t,u) = \frac{4g_\pi^4 \delta(1-\delta)}{Z_\pi^2 m(m-\widehat{m})} \Bigg[\left(\frac{2\widehat{m}-m}{m-\widehat{m}}\right)(3s-4m_\pi^2) I_2(m_\pi^2) \\ -D(s,t,u) + 2\delta C(s,t,u)\Bigg].\end{aligned} \tag{128}$$

$$\begin{aligned}\Delta_\sigma(s,t,u) = \frac{4g_\pi^4}{Z_\pi^2(m-\widehat{m})^2} \Big\{ m_\pi^2 \delta I_2(m_\pi^2) + (4m^2 - s) I_2(s) \\ -4m(m-\widehat{m}) \left[2\left(1 - \frac{s(1-\delta^2)}{4m^2}\right) I_2(s) + \frac{m_\pi^2}{m^2}\delta(1-\delta) I_2(m_\pi^2) \right. \\ \left. +\delta^2(s-2m_\pi^2) I_3(q_1,-q_2)\right]\Big\}.\end{aligned} \tag{129}$$

$$\begin{aligned}\Delta_{rest}(s,t,u) = \frac{4g_\pi^4}{Z_\pi^2} \Bigg\{ \left[\frac{s + 4m(m-2\widehat{m})}{(m-\widehat{m})^2} - \frac{2s(1-\delta^2)}{m(m-\widehat{m})} \right] I_2(s) \\ +\frac{m_\pi^2 \delta}{m(m-\widehat{m})^2} \left[4\widehat{m}(1-\delta) - m + \frac{3s}{m_\pi^2}(1-\delta)(m-2\widehat{m}) \right] I_2(m_\pi^2) \\ +\frac{4m\delta^2}{(m-\widehat{m})}(s-2m_\pi^2) I_3(q_1,-q_2) \\ +\frac{\delta(1-\delta)}{m(m-\widehat{m})} \left[D(s,t,u) - 2\delta C(s,t,u)\right] \Bigg\}.\end{aligned} \tag{130}$$

Here the notation

$$D(s,t,u) = \frac{(1-\delta)}{6\delta m^2} \left(\delta + \frac{\widehat{m}}{m-\widehat{m}}\right) \left[(s-u)tJ_2(t) + (s-t)uJ_2(u)\right] \tag{131}$$

has been used. Summing (128)-(130) it is easy to see that

$$\Delta_\rho(s,t,u) + \Delta_\sigma(s,t,u) + \Delta_{rest}(s,t,u) = 0. \tag{132}$$

This is what we wanted to show.

3.4 The full result

Now we can go on and estimate the role of higher order contributions in the amplitude obtained. For direct comparison with the empirical numbers we fix parameters in order to obtain the pion decay constant, f_π, and the pion mass, m_π, close to their physical values, $f_\pi \simeq 93$ MeV and $m_\pi \simeq 139$ MeV, respectively. We take two sets of parameters, set I with a constituent quark mass $m \simeq m_\rho/2$, set II with a rather low quark mass. In the first case we get a bound state for the ρ - meson, in the other case the ρ - meson mass lies above the threshold for production of a pair of 'free' quarks, a problem related to the absence of confinement of the model. Nevertheless, as will be discussed in the next section, a better description of the scalar form factor of the pion is only achieved for rather low constituent quark masses, so that, with due care, it is of interest to study pion observables for this case too. The model parameter G_V is obtained by fixing the mass of the ρ meson, $m_\rho \simeq 770$ MeV, as long as $m_\rho < 2m$. For the low constituent mass case, the ρ is embedded deeply in the $\bar{q}q$ continuum. To avoid the complications of defining this isovector-vector state under such circumstances [27], we choose to fit the scattering length a_1^1 to fix G_V. This is an unambiguous and rather simple procedure. As a result, the four parameters of the model for set I have the values: $G_S = 9.41\,\mathrm{GeV}^{-2}$, $G_V = 11.29\,\mathrm{GeV}^{-2}$, $m = 390\,\mathrm{MeV}$ ($\widehat{m} = 3.9$ MeV) and $\Lambda = 1$ GeV. With these parameters, we find $f_\pi = 92\,\mathrm{MeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\rho = 770\,\mathrm{MeV}$, $\delta = 0.62$. For set II we obtain $G_S = 1.083\,\mathrm{GeV}^{-2}$, $G_V = 8.8\,\mathrm{GeV}^{-2}$, $m = 200\,\mathrm{MeV}$ ($\widehat{m} = 1.0\,\mathrm{MeV}$), $\Lambda = 2.5$ GeV, $f_\pi = 92.7\,\mathrm{MeV}$, $m_\pi = 139\,\mathrm{MeV}$, $\delta = 0.69$. For obvious reasons, phase shifts are always presented for $\sqrt{s} < 2m$. We also calculate the set of parameters in the limiting case $G_V \to 0$ or $\delta \to 1$, already determined in [2]. Here we use $G_S = 7.74\,\mathrm{GeV}^{-2}$, $\Lambda = 1\,\mathrm{GeV}$ and $m = 242\,\mathrm{MeV}$ ($\widehat{m} = 5.5\,\mathrm{MeV}$). Therefore one can always compare the predictions of the NJL and extended NJL models. Apart from that, we need values of the main physical quantities in the chiral limit $\widehat{m} \to 0$. In this case we obtain for set I, $\overset{\circ}{m} = 382\,\mathrm{MeV}$, $f = 90.9\,\mathrm{MeV}$, $\overset{\circ}{\delta} = 0.628$ and for set II, $\overset{\circ}{m} = 183\,\mathrm{MeV}$, $f = 88.4\,\mathrm{MeV}$, $\overset{\circ}{\delta} = 0.73$ The leading term of the chiral expansion for the pion mass is $\overset{\circ}{m}_\pi = 138.42$ MeV in I and $\overset{\circ}{m}_\pi = 141$ MeV in set II.

First, we give the values for the constants L_2 and L_3: we have for set I, $L_2 = 1.2 \cdot 10^{-3}$, $L_3 = -3.2 \cdot 10^{-3}$ and for set II, $L_2 = 2.0 \cdot 10^{-3}$, $L_3 = -2.2 \cdot 10^{-3}$. For comparison, we give the scale–dependent empirical values $L_2(m_\rho) = 1.2 \cdot 10^{-3}$, $L_3(m_\rho) = -3.6 \cdot 10^{-3}$. However such a comparison has to be taken cum grano salis since the L_i calculated within the Hartree approximation are simple c-numbers. Whereas set I lead to values closer to the experimental ones, set II is compatible with the results of [14]. Second, we consider the $\pi\pi$ threshold parameters. The results of our calculations are presented in the Table. The first and second columns list the numbers based on the chiral expansion (E^4 approximation), for sets I and II. The formulae used are given in the Appendix. The third and fourth columns show the results of the exact calculations on the basis of the full amplitudes $A(s,t,u)$, for I and II. We compare them with the results from the standard NJL approach without vector mesons [2], within the current algebra [16], and with the experimental data [28].

a_ℓ^I	$\mathcal{O}(E^4)[I]$	$\mathcal{O}(E^4)[II]$	total [I]	total [II]	NJL	SMT	exp.
a_0^0	0.16	0.19	0.17	0.19	0.19	0.16	0.26 ± 0.05
b_0^0	0.18	0.25	0.19	0.25	0.27	0.18	0.25 ± 0.03
a_0^2	-0.046	-0.043	-0.047	-0.045	-0.044	-0.045	-0.028 ± 0.012
b_0^2	-0.088	-0.078	-0.090	-0.079	-0.079	-0.089	-0.082 ± 0.008
a_1^1	0.034	0.037	0.038	0.039^*	0.034	0.030	0.038 ± 0.002
$a_2^0 \times 10^4$	5.9	21.0	6.9	18.5	16.7		17 ± 3
$a_2^2 \times 10^4$	-2.1	4.7	-2.5	0.0	3.2		1.3 ± 3

Table: The $\pi\pi$ scattering lengths and effective ranges in the extended NJL model in comparison with the same calculations in the original NJL model, without spin-one mesons [2], and the soft meson theorems (SMT) [16]. The experimental data are taken from [28]. The '*' denotes an input quantity.

From this comparison one infers the following. All scattering lengths and range parameters are mainly determined by the $\mathcal{O}(E^4)$ approximation. The effect of higher-order terms is noticeable only in the D-waves. If we compare the results of numerical calculations with our previous estimations of low-energy $\pi\pi$ scattering parameters within the NJL model without vector mesons, we notice that the agreement with the experimental data is of comparable quality (for set II) but poorer for set I. This confirms the observation of Ref.[29] that within the Hartree approximation one needs to work with a small constituent quark mass. Similar findings were obtained in Ref.[14]. Alternatively, one could think of going beyond the Hartree approximation and, in particular, include pion loops. This is a difficult problem which deserves further studies [30].

In Fig.6a,b we show the S and P-wave phase shifts δ_0^0, δ_0^2 and δ_1^1 for set I in comparison with the available data ($\sqrt{s} \leq 700$ MeV). The calculated phases are in reasonable agreement with the data. For set II, we are confined to $\sqrt{s} \leq 400$ MeV and therefore only show the S-wave phases in Fig.6c. New data in this region will come once DAΦNE is operating and $K_{\ell 4}$ decays have been analyzed.

4 The scalar form factor of the pion

Another quantity closely related to $\pi\pi$ scattering is the scalar form factor of the pion, defined by

$$< \pi^a(p') \mid \widehat{m}(u\bar{u} + d\bar{d}) \mid \pi^b(p) >= \delta^{ab}\Gamma_\pi(t) \tag{133}$$

where $t = (p - p')^2$ is the square of the invariant four-momentum transfer. In the framework of the original NJL model this quantity was previously determined [34] and agreed quite well with the empirical scalar form factor of the pion for low momentum transfers at a fairly small constituent quark mass of 241.8 MeV. In particular, the scalar pion radius was demonstrated to impose powerful constraints on the parameters of the NJL model. We anticipate that with the large constituent mass as in set I, the scalar radius will be largely underestimated. The relevant Feynman diagrams to calculate this quantity are

depicted in Fig.7. There is a sum of two contributions to the scalar pion form factor in the NJL model: one is the direct coupling of the operator $\widehat{m}(u\bar{u}+d\bar{d})$ (double-dashed line) to the two pions via a quark triangle (we call it bare coupling), and the other corresponds to rescattering of quarks into the scalar meson σ, which in turn couples to the pions. In Fig.7a we show these contributions, with the pion legs amputated. After solving the Bethe–Salpeter equation for this vertex, Γ, one obtains the scalar pion form factor by attaching the pion legs, Fig.7b. With eqs. (20) and (98), the result for the scalar form factor of the pion $\Gamma_\pi(t)$ can be cast in the form

$$\Gamma_\pi(t) = \frac{\widehat{m}(4g_\pi)^2}{Z_\pi}\left[1+\frac{\mathcal{J}_s(t)g^2(t)}{m_\sigma^2(t)-t}\right]\mathcal{K}(t) \tag{134}$$

where

$$\mathcal{K}(t) = \left[m-\frac{t\beta_\pi}{2}(1+\delta)\right]I_2(t) + m_\pi^2\beta_\pi\delta I_2(m_\pi^2) + \frac{m}{2}\delta^2(t-2m_\pi^2)I_3(p,p'), \tag{135}$$

and $\mathcal{J}_s(t) = 8I_1 + 4(t-4m^2)I_2(t)$ is the fundamental quark bubble in the scalar channel. For the same parameter sets used in the evaluation of the scattering parameters, we obtain $\Gamma_\pi(0) = 1.007m_\pi^2$, (set I), and $\Gamma_\pi(0) = 0.946m_\pi^2$, (set II), consistent with the χPT prediction [23]. By expanding in powers of t we extract the scalar mean square radius of the pion

$$\frac{\Gamma_\pi(t)}{\Gamma_\pi(0)} = 1 + \frac{1}{6} < r^2 >_\pi^s t + \mathcal{O}(t^2). \tag{136}$$

Its numerical value $< r^2 >_\pi^s = 0.043\,\mathrm{fm}^2$ calculated with the parameter set I is one order of magnitude smaller than the empirical value $< r^2 >_\pi^s = (0.55\pm0.15)\,\mathrm{fm}^2$ [29] extracted from phase shift analyses [35]. For set II we increase the mean scalar radius $< r^2 >_\pi^s = 0.53\,\mathrm{fm}^2$ in good agreement with the empirical number. To understand these numbers in more detail, we investigate the chiral expansion of the scalar form factor up-to-and-including terms of order E^4,

$$\Gamma_\pi(0) = \overset{\circ}{m}{}_\pi^2\left\{1+\frac{\overset{\circ}{m}{}_\pi^2\overset{\circ}{\delta}}{2\overset{\circ}{m}{}^2}\left[1-2\overset{\circ}{\delta}-\frac{\overset{\circ}{m}{}^2\overset{\circ}{h}_1\overset{\circ}{\delta}(1-3\overset{\circ}{\delta})}{2\pi^2f^2}\right]\right\} + \mathcal{O}(E^6). \tag{137}$$

In the leading order of the chiral expansion we get exactly the current algebra result $\Gamma_\pi(0) = \overset{\circ}{m}{}_\pi^2$ [36], as expected, since all symmetries are respected in the evaluation of the form factor. The E^4 contribution leads to $\Gamma_\pi(0) = 1.005\ \overset{\circ}{m}{}_\pi^2$ for set I and $\Gamma_\pi(0) = 0.943\ \overset{\circ}{m}{}_\pi^2$ in set II. For the scalar radius of the pion we have

$$< r^2 >_\pi^s = \frac{3\overset{\circ}{\delta}{}^2}{2\overset{\circ}{m}{}^2}\left[1-\frac{3\overset{\circ}{m}{}^2\overset{\circ}{\delta}\overset{\circ}{h}_1}{(2\pi f)^2}\right] \tag{138}$$

which is $\approx 0.057\,\mathrm{fm}^2$ in set I and $\approx 0.70\,\mathrm{fm}^2$ in set II, i.e. in this quantity one observes large higher order effects. This is not unexpected from previous calculations in chiral

perturbation theory beyond one loop [35]. These higher order effects are essentially accounted for by the low constituent quark mass in agreement with the findings of Ref.[29]. We stress again that alternatively one might want to consider pion loop effects, which, however, goes beyond the scope of the present manuscript.

5 Summary and conclusions

In the present paper we have developped a method which allows for the calculation of N-point functions within the extended Nambu–Jona-Lasinio model with nonlinear realization of chiral symmetry [14]. This extends the ideas described in [2]. Using the path integral technique and the Hartree approximation, we work in momentum space to circumvent the standard way of employing the heat kernel expansion for constructing an effective meson Lagrangian. With the heat kernel method, one manages to get only the first few terms of the Lagrangian (in general the $\mathcal{O}(E^4)$ approximation) and faces growing difficulties when calculating terms of yet higher orders. Examining the one-loop approximation for the effective action in momentum space, we find transformations of collective variables which permit to calculate these higher order terms in a straightforward manner. The two-point functions describing propagation of collective excitations lead to results coinciding with those obtained from analysis of Bethe–Salpeter equations on bound quark-antiquark states for two-particle Green functions of quark fields.

Using equations for the mass of constituent quarks (gap equation) and the pion mass, we construct chiral expansions for these quantities and the constant f_π. We have also calculated the $\pi\pi$ scattering amplitude. This program is applied both to the linear and nonlinear description of transformation properties of physical fields. We have demonstrated explicitely that the $\pi\pi$ scattering amplitude $A(s,t,u)$ obtained in both approaches is completely equivalent. We have carried out the chiral expansion for this amplitude and calculated the pertinent scattering lengths and range parameters. In addition we have obtained the corresponding low-energy $\pi\pi$ phase shifts. All these calculations have been performed for two sets of parameters, the first and the second one having a large (~ 400 MeV) and a small (~ 200 MeV) constituent mass, respectively. In general, the conclusion is that the corrections derived by considering terms of the effective meson Lagrangian with higher derivatives are small for S– and P–waves but are significant in the D–waves. Agreement with the experimental data is of comparable quality than in the model ignoring vector modes for the low constituent quark mass case. This conclusion has been sharpened further by studying the scalar pion radius. Only for low constituent masses one can describe this quantity within the Hartree approximation of the (extended) NJL model [29]. Such low constituent quark masses are also obtained in the currently popular estimates of the chiral perturbation theory low-energy constants from extended NJL models [14]. Further research in this direction should be concerned with a consistent implementation of pion loop effects (for some first attempts see e.g. [30]).

APPENDIX

Here we give expressions for the main low-energy characteristics of the $\pi\pi$ scattering derived in the $\mathcal{O}(q^4)$ approximation from the amplitude (105) + (111). They coincide with the results of [20].

$$a_0^0 = \frac{7\,\mathring{m}_\pi^{\,2}}{32\pi f^2}\left[1 + \frac{5\,\mathring{m}_\pi^{\,2}}{84\pi^2 f^2}\left(\bar{l}_1 + 2\bar{l}_2 - \frac{9\bar{l}_3}{10}\right)\right]. \tag{139}$$

$$a_0^2 = -\frac{\mathring{m}_\pi^{\,2}}{16\pi f^2}\left[1 - \frac{\mathring{m}_\pi^{\,2}}{12\pi^2 f^2}\left(\bar{l}_1 + 2\bar{l}_2\right)\right]. \tag{140}$$

$$\mathring{m}_\pi^{\,2}\, b_0^0 = \frac{\mathring{m}_\pi^{\,2}}{4\pi f^2}\left[1 + \frac{\mathring{m}_\pi^{\,2}}{12\pi^2 f^2}\left(2\bar{l}_1 + 3\bar{l}_2\right)\right]. \tag{141}$$

$$\mathring{m}_\pi^{\,2}\, b_0^2 = -\frac{\mathring{m}_\pi^{\,2}}{8\pi f^2}\left[1 - \frac{\mathring{m}_\pi^{\,2}}{12\pi^2 f^2}\left(\bar{l}_1 + 3\bar{l}_2\right)\right]. \tag{142}$$

$$\mathring{m}_\pi^{\,2}\, a_1^1 = \frac{\mathring{m}_\pi^{\,2}}{24\pi f^2}\left[1 + \frac{\mathring{m}_\pi^{\,2}}{12\pi^2 f^2}\left(\bar{l}_2 - \bar{l}_1\right)\right]. \tag{143}$$

$$\mathring{m}_\pi^{\,4}\, a_2^0 = \frac{\mathring{m}_\pi^{\,4}}{1440\pi^3 f^4}\left(\bar{l}_1 + 4\bar{l}_2\right). \tag{144}$$

$$\mathring{m}_\pi^{\,4}\, a_2^2 = \frac{\mathring{m}_\pi^{\,4}}{1440\pi^3 f^4}\left(\bar{l}_1 + \bar{l}_2\right). \tag{145}$$

The quantity $\bar{l}_3$ appears in the chiral expansion for the pion mass

$$m_\pi^2 = \mathring{m}_\pi^{\,2}\left(1 - \frac{\mathring{m}_\pi^{\,2}\,\bar{l}_3}{32\pi^2 f^2}\right). \tag{146}$$

It is

$$\bar{l}_3 = 4\,\mathring{\delta}\left[\mathring{\delta}\,(1 - 3\,\mathring{\delta})\,\mathring{h}_1 + \frac{2\pi^2 f^2}{\mathring{m}^{\,2}}(2\,\mathring{\delta} - 1)\right]. \tag{147}$$

For the pion decay constant one can obtain

$$f_\pi = f\left(1 + \frac{\mathring{m}_\pi^{\,2}}{(4\pi f)^2}\bar{l}_4\right), \tag{148}$$

where

$$\bar{l}_4 = \mathring{\delta}^2\left[\left(\frac{2\pi f}{\mathring{m}}\right)^2 - 3\,\mathring{\delta}\mathring{h}_1\right]. \tag{149}$$

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Figure captions

Fig.1. One-loop quark diagrams corresponding to the amplitude $A(s,t,u)$. (a) ρ meson exchange; (b) scalar particle exchange; (c) other contributions. The structure of meson vertices is shown in other figures.

Fig.2. Vertices $\rho \to \pi\pi$ and $\sigma \to \pi\pi$ in the model with linear realization of chiral symmetry.

Fig.3. Diagrams corresponding to vertex (c) (see Fig.1) in the model with linear realization of chiral symmetry.

Fig.4. Vertices $\rho \to \pi\pi$ and $\sigma \to \pi\pi$ in the model with nonlinear realization of chiral symmetry. Added to these diagrams should be effects of πa_1 mixing on lines with ξ_μ.

Fig.5. Diagrams corresponding to vertex (c) (see Fig.1) in the model with nonlinear realization of chiral symmetry. Added to these diagrams should be effects of πa_1 mixing on lines with vertex ξ_μ (with the one pion field only).

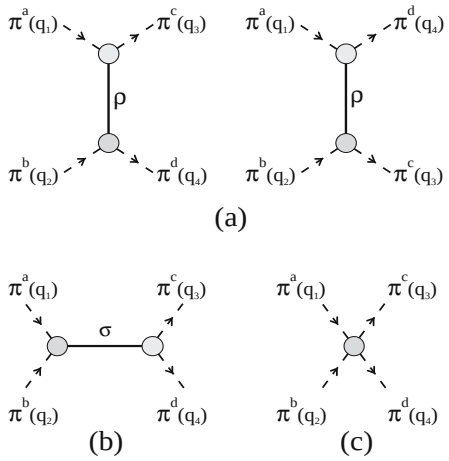
Fig.6a. The S-wave phase shifts for the heavy mass case (in degrees). Upper panel: δ_0^0, lower panel: δ_0^2. Data for δ_0^0 are from Refs.[31] and for δ_0^2 from Refs.[32].

Fig.6b. The P-wave phase shift for the heavy mass case (in degrees). Data are from Refs.[33].

Fig.6c. The S-wave phase shifts for the low mass case (in degress). Upper panel: δ_0^0, lower panel: δ_0^2.

Fig.7a. The full scalar quark form factor, Γ, is the sum of the bare coupling of the operator $\widehat{m}(u\bar{u} + d\bar{d})$ (double line) to the quarks and a contribution from rescattering of the quarks through the four-fermion interaction in the scalar channel, with strength G_S. The latter can be represented as a coupling to the composite meson σ.

Fig.7b. The scalar pion form factor Γ_π obtained by attaching two pion legs to Γ.


π^a(q1)
π^c(q3)
ρ
π^b(q2)
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(a)
π^a(q1)
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ρ
π^b(q2)
π^c(q3)
π^a(q1)
π^c(q3)
σ
π^b(q2)
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(b)
π^a(q1)
π^c(q3)
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π^d(q4)
(c)

Figure 1

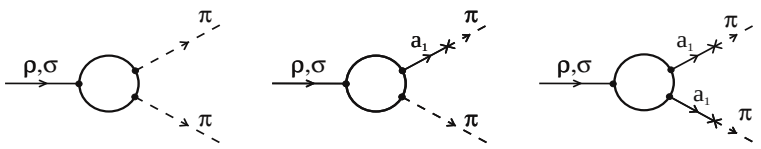


Figure 2

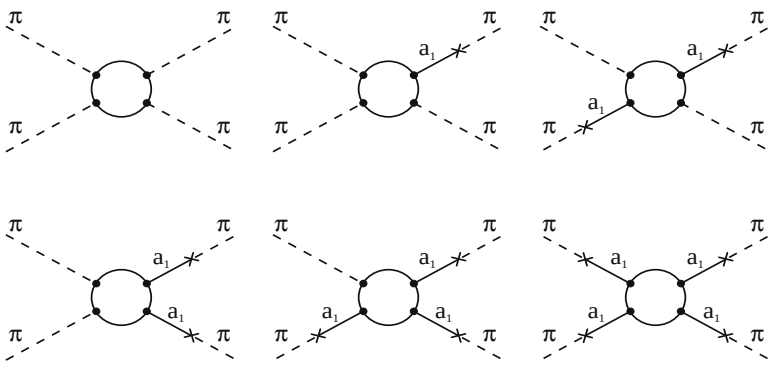


Figure 3

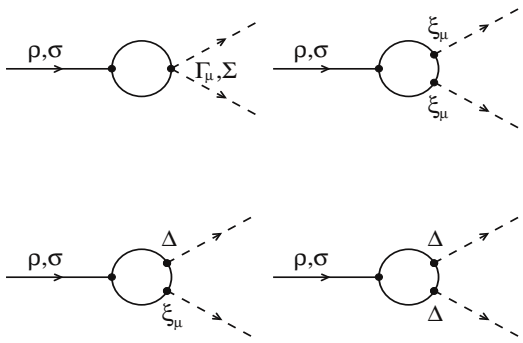
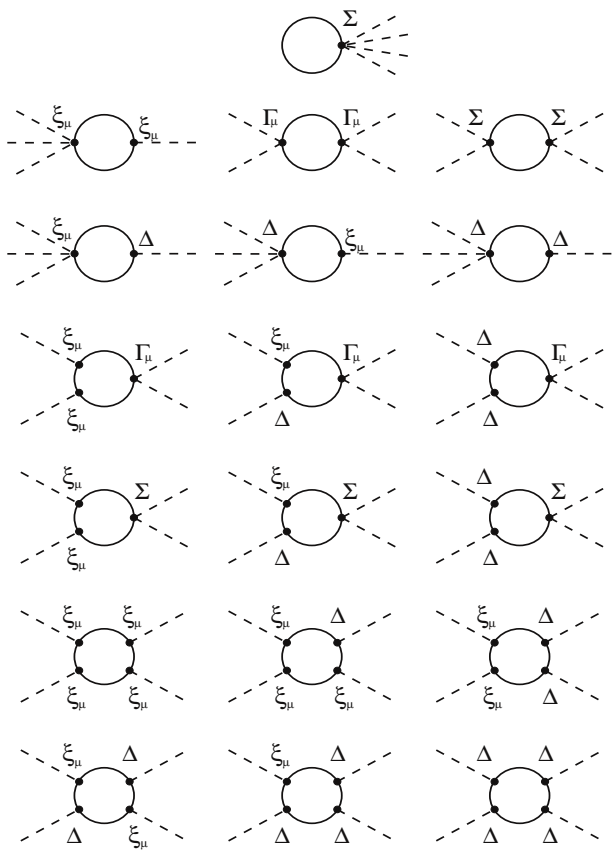


Figure 4


Σ
ξμ ξμ
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ξμ Δ Δ Δ
Δ Δ Δ Δ

Figure 5

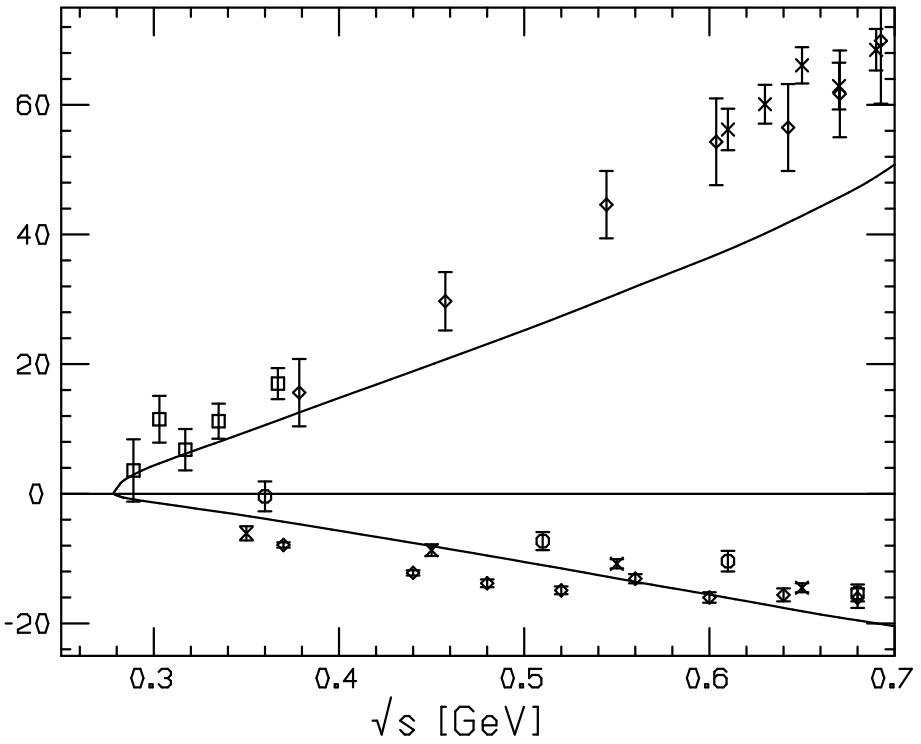

Figure 6a

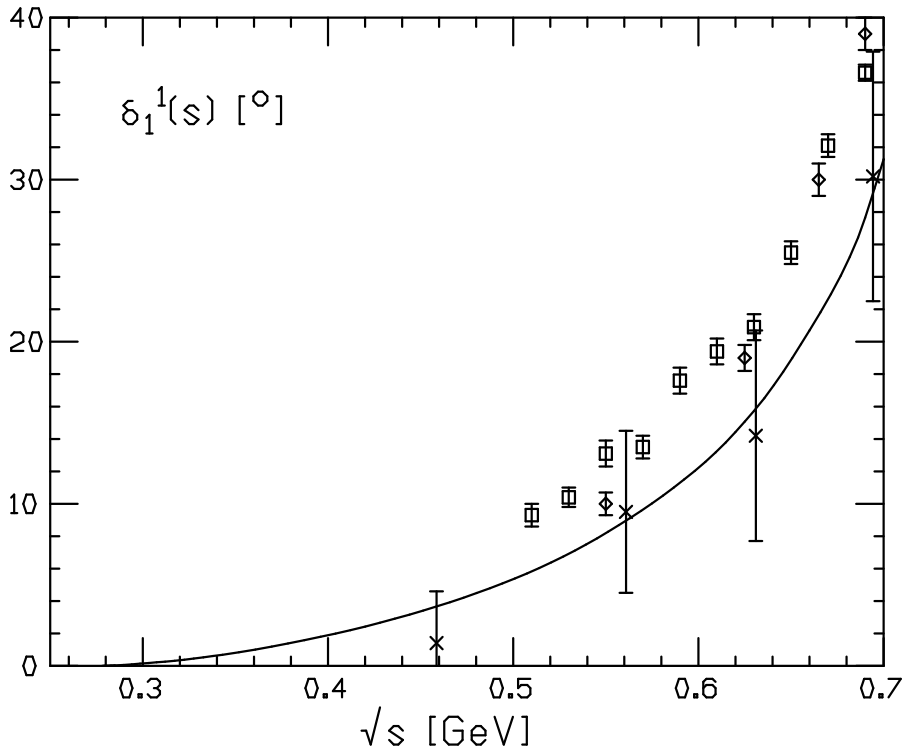

Figure 6b

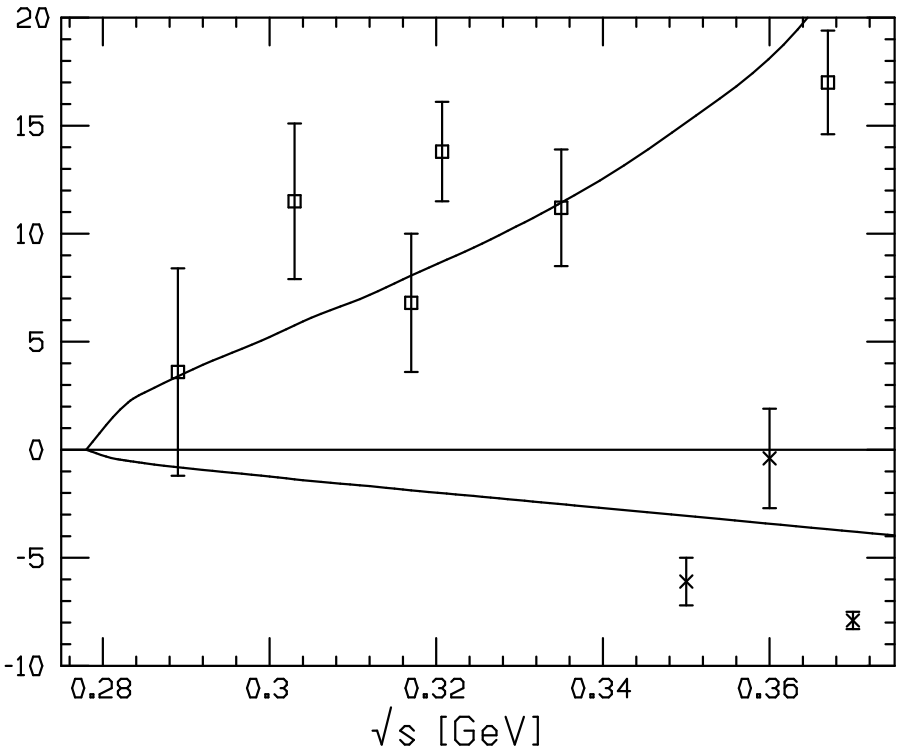

Figure 6c

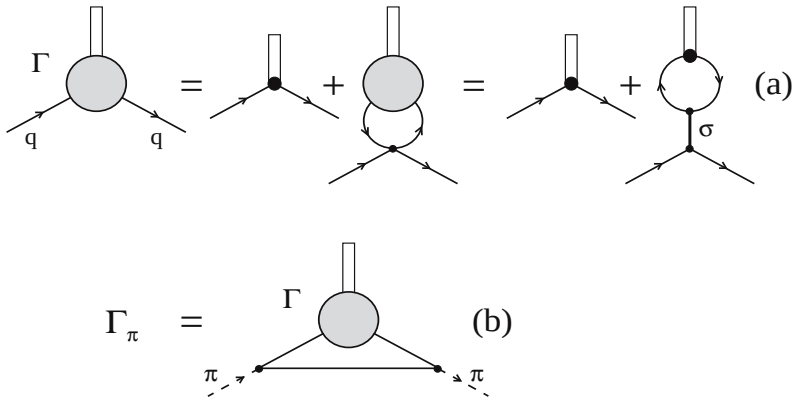

Figure 7